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On Processibility by Lemke's Algorithm of Schultz's LCP Formulation of Switching Control Stochastic Games

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Abstract Schultz (1992) formulated 2-person, zero-sum, discounted switching control stochastic games as a Linear Complementarity Problem (LCP) and discussed computational results. It remained open to prove or disprove Lemke-processibility of this LCP. We show that Lemke's algorithm does not always sucessfully process this LCP.

We propose two new LCP formulations with the aim of making the underlying matrix belong to the classes R_0 and E_0 which would imply Lemke processibility. While, formulating switching control games as an E_0-matrix LCP leads to the violation of certain constraints, we prove that the underlying matrix in one of these formulations is an R_0-matrix.

Keywords Discounted Switching Control Stochastic Games, Linear Complementarity Problem (LCP), Processibility by Lemke's Algorithm, Secondary Ray Termination, E_0 property (or the class E_0), R_0 property (or the class R_0).

1 Introduction

The problem of computing the value and optimal strategies in various classes of stochastic games has been well studied (Mohan et al (2001), Nowak and Raghavan (1993), Raghavan and Syed (1981), Parthasarathy et al (1984), Raghavan (2003), Raghavan and Syed (2002), Sobel (1981)), and continues to draw significant interest.

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Solving some of these classes of stochastic games by formulating them as a linear complementarity problem (LCP) has been studied as well (Mohan et al (2001), Raghavan and Syed (2002), Schultz (1992)).

Given a square matrix M of order n and a vector q of length n, the problem $LCP(q,M)$ is to find w≥0, z ≥0 such that $w - Mz = q, w'z = 0$. (w,z) is said to be a solution of $LCP(q,M)$ (if such a solution exists).

Lemke (1965) proposed a complementary pivot algorithm to solve LCPs. Lemke's algorithm either finds a solution (degenerate or otherwise) or terminates in a secondary ray, in which case the algorithm cannot proceed further. Significant research has been done (for example, Garcia (1973)) to identify classes of matrices, such as $E_0 \cap R_0$, for which Lemke's algorithm finds a solution (and does not terminate in a ray). We define these concepts in subsequent sections.

Schultz (1992) proposed an LCP formulation for discounted zero-sum switching control stochastic games and discussed computational results, namely, that Lemke's algorithm found a solution for 100 randomly generated games. It remained open either to prove that Schultz's formulation is processible by Lemke's algorithm or to provide a counter example. We show that Lemke's algorithm does not always sucessfully process this LCP, even when initialized as done in Schultz (1992). We suggest alternative formulations, the underlying matrix in one of which is R_0.

2 Background and Preliminaries

2.1 Stochastic Games (Shapley, 1953)

2.1.1 Zero-Sum Two-Player Stochastic Games

We define a zero-sum, 2-player, finite states space, finite action space stochastic game as consisting of the following:

1. A finite set of states, $S = \{1, 2, ..., N\}$.
2. $A(s) = \{1, 2, ..., m_s\}$ and $B(s) = \{1, 2, ..., n_s\}$, finite sets of actions for players 1 and 2 respectively, for each state $s \in S$.
3. For each state $s \in S$, a matrix of immediate rewards to player 1 (from player 2), $R(s) = [r(s,a,b)]$.
4. Transition probabilities $Q_{a,b}(s,t) = [q(t|s,a,b)]$ where $q(t|s,a,b)$ is the probability of transition from state s to state t given that players 1 and 2 choose actions $a \in A(s)$, $b \in B(s)$ respectively.

The game proceeds as follows. Given a starting state $s_0 \in S$, the players choose actions $a_0 \in A(s_0)$ and $b_0 \in B(s_0)$ and player 2 pays $r(s_0, a_0, b_0)$ to player 1. The game moves to a new state s_1 according to $Q_{a_0,b_0}(s_0, s)$ and the game continues infinitely. Strategies as mentioned above, where each player picks an action at every state, are called pure strategies. On the other hand, the players can also choose to play probability distributions over their respective action sets. Such strategies are called mixed strategies. We shall only consider *stationary strategies* which are strategies that depend only on the current state s and not on how s was reached.

Given strategies f, g and an initial state s_0, we define the β-discounted payoff

$$[I_\beta(f, g)](s_0) = \sum_{t=0}^{\infty} \beta^t\ r_t(s_0, f, g) \text{ for a discount factor } \beta \in [0, 1)$$

and the undiscounted (or limiting average) payoff

$$[\phi(f, g)](s_0) = \liminf_{T\uparrow\infty} \frac{1}{T+1} \sum_{t=0}^{T} r_t(s_0, f, g).$$

In this paper, we shall be looking at the case of discounted payoffs. In this case, a pair of strategies *(f*, g*)* is optimal, if for all $s \in S$

$$[I_\beta(f, g^*)](s) \leq [I_\beta(f^*, g^*)](s) \leq [I_\beta(f^*, g)](s),$$

for all pairs of strategies (f, g) of players 1 and 2.

2.1.2 Switching Control Stochastic Games

In a switching control stochastic game, player 1 controls the transitions in certain states and player 2 in the other states. That is,

$$q(s' \mid s,a,b) = q(s' \mid s,a), \text{ for } s' \in S, s \in S_1, a \in A(s) \text{ and } \forall b \in B(s),$$

$$q(s' \mid s,a,b) = q(s' \mid s,b), \text{ for } s' \in S, s \in S_2, b \in B(s) \text{ and } \forall a \in A(s).$$

For the sake of completeness, we state the following theorem from Filar et al (1991), restated in Schultz (1992).

Notation: In the following, 0_k stands for the k-dimensional column vector of 0's, $0_{k_1 \times k_2}$ stands for the matrix of all 0's of order $k_1 \times k_2$ and e_k stands for the k-dimensional column vector of 1's.

Theorem 2.1 *: A β-discounted zero-sum stochastic game possesses value $v_\beta(s)$ for $s \in S$ and optimal stationary strategies x and y for players* 1 *and* 2 *respectively if and only if*

$$x \in X_s, y \in Y_s \tag{2.1}$$

$$v_\beta(s)e_{m_s} - \beta \sum_{t \in S} Q(s,t)y(s) - R(s)y(s) \geq \mathbf{0_{m_s}}, \ \ \forall\, \mathbf{s} \in \mathbf{S}, \tag{2.2}$$

$$-v_\beta(s)e_{n_s} + \beta \sum_{t \in S} v_\beta(t)[x(s)' Q(s,t)]' + [x(s)' R(s)]' \geq \mathbf{0_{n_s}}, \ \ \forall\, \mathbf{s} \in \mathbf{S}. \tag{2.3}$$

It follows that if $v_\beta(s)$, $x(s)$, $y(s)$ satisfy (2.1), (2.2) and (2.3) then

$$v_\beta(s) = \beta \sum_{t \in S} v_\beta(t) x(s)' Q(s,t) y(s) + [x(s)' R(s)] y(s), \ \ \forall\, s \in S. \tag{2.4}$$

For β-discounted zero-sum switching control stochastic games, the above theorem can be rewritten as follows.

Theorem 2.2 *: A β-discounted zero-sum switching control stochastic game possesses values $v_\beta(s)$ for $s \in S$ and optimal stationary strategies $x(s)$ and $y(s)$ if and only if*

$$x \in X_s, y \in Y_s \tag{2.5}$$

$$v_\beta(s)e_{m_s} - \beta \sum_{t\in S} v_\beta(t)q^1(s,t) - R(s)y(s) \geq \mathbf{0_{m_s}}, \;\; \forall\, \mathbf{s} \in \mathbf{S_1}, \tag{2.6}$$

$$(v_\beta(s) - \theta_\beta(s))e_{m_s} - R(s)y(s) \geq \mathbf{0_{m_s}}, \;\; \forall\, \mathbf{s} \in \mathbf{S_2}, \tag{2.7}$$

$$(-v_\beta(s) + \theta_\beta(s))e_{n_s} - (x(s)R(s))' \geq \mathbf{0_{n_s}}, \;\; \forall\, \mathbf{s} \in \mathbf{S_1}, \tag{2.8}$$

$$-v_\beta(s)e_{n_s} + \beta \sum_{t\in S} v_\beta(t)q^2(s,t) + [x(s)'R(s)]' \geq \mathbf{0_{n_s}}, \;\; \forall\, \mathbf{s} \in \mathbf{S_2}. \tag{2.9}$$

$$x(s) \textit{ is complementary in (2.5) and (2.6)} \tag{2.10}$$

$$y(s) \textit{ is complementary in (2.8) and (2.9)} \tag{2.11}$$

where $\theta_\beta(s) = \begin{cases} \beta \sum_{t\in S} v_\beta(s)x(s)'q^1(s,t), \textit{ for } s \in S_1 \\ \beta \sum_{t\in S} v_\beta(s)q^2(s,t)'y(s), \textit{ for } s \in S_2 \end{cases}$

2.2 Processibility of LCPs by Lemke's algorithm

We use the implementation of Lemke's algorithm as discussed by Murty (1997).

The Class E_0: $M \in E_0$, if $\forall 0 \neq z \geq 0, \exists i$ such that $z_i > 0$ and $(Mz)_i \geq 0$.

The Class R_0: $M \in R_0$, if $LCP(0,M)$ has a unique solution, viz., $(0,0)$.

Theorem 2.3 *: (Sufficient condition for Lemke processibility): $LCP(q,M)$ is processible by Lemke's algorithm if M is both E_0 and R_0.*

Apart from the book by Murty (1997), the reader is also encouraged to refer to the paper by Cottle and Dantzig (1968) and the book by Cottle et al (1992) for an excellent treatment of the Linear Complementarity Problem, and to (Mohan et al, 1996) for proofs of equivalence of the LCP and the General LCP.

3 On Lemke-Processibility of Schultz's LCP Formulation

3.1 Schultz's Formulation

For the sake of completeness, we describe Schultz's formulation ((Schultz, 1992)) below. Using theorem 2.2, Schultz (1992) formulated 2-player, Zero-Sum, Switching Control Stochastic Games as the following Linear Complementarity Problem:
$w - Mz = q = -c$ where $M = \begin{bmatrix} R & B \\ A & 0 \end{bmatrix}$ is a $K \times K$ matrix, K = 4N + (m + n), m = $\sum_{s\in S} m_s$, n = $\sum_{s\in S} n_s$.

$$z' = \begin{bmatrix} x' & y' \end{bmatrix} \text{ where } x = \begin{bmatrix} v_1 \\ v_2 \\ \theta_1 \\ \theta_2 \end{bmatrix}, y = \begin{bmatrix} f(1) \\ \vdots \\ f(N) \\ g(1) \\ \vdots \\ g(N) \end{bmatrix},$$

v and θ are unbounded variables and are written as the difference of non-negative variables as follows: $v(s) = v_1(s) - v_2(s), \theta(s) = \theta_1(s) - \theta_2(s)$
B and R are defined so that $Bx + Ry \geq 0$ is equivalent to (2.6) through (2.9). A is defined to make f and g probability vectors.

3.2 Examples where Lemke's Algorithm Terminates in a Secondary Ray

Schultz (1992) discusses formulation, initialization and processing of the following switching control stochastic game. In this game, $N = 2, m_1 = m_2 = n_1 = n_2 = 2$ and following are the rewards and transitions:

$$R(1) = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, q^1(1,1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, q^1(1,2) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$R(2) = \begin{bmatrix} 1 & 4 \\ 5 & 1 \end{bmatrix}, q^2(2,1) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, q^2(2,2) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

Discount factor $\beta = 0.8$.

3.2.1 Paths in the example in Schultz (1992) that lead to a secondary ray

Starting with the initial tableau as given below, we find that, even if we initialize d = c as suggested in Schultz (1992), there are paths that lead to a secondary ray and hence, Lemke's algorithm cannot proceed further.

		z	w	z0		q
w		$\begin{matrix} 0 & -A \\ -B & -R \end{matrix}$	I	-d		-c

In our program, we used the lexicographic method to choose the row corresponding to the minimum ratio and as described in Murty (1997), this technique ensures a unique row in each iteration. We also experimented with other methods such as "always choosing the first minimum row", "always choosing the last minimum row" and so on and in all our experiments, the algorithm reached a secondary ray. Of course, there are "good paths" too - paths that lead to a solution (degenerate or otherwise).

3.2.2 1-state example

We consider the following example where there is just 1 state (say, controlled by player 1), each player has just 1 action and (obviously), the state is an absorbing state. The reward to player 1 is 1 (say). Here, again, one of the paths in Lemke's algorithm leads to a secondary ray.

The matrices are as follows:

$$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.2 & -0.2 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

We list the tables and discuss further details in appendix 1.

For both the above examples, we experimented with different initializations of d, for example $d = q = -c, d = -q = c, d = e$, the vector of all 1's, and d with each entry unique, and for all these cases, Lemke's algorithm terminates in a secondary ray. As discussed by Kostreva (refer the book by (Murty, 1997)), successful termination of Lemke's algorithm heavily depends on the choice of d. Murty (1997) discusses some examples where Lemke's algorithm finds a solution for some choice of d and ends up in a secondary ray for a different choice of d. In appendix 1, we discuss a choice of d for the 1-state example, for which Lemke's algorithm always finds a solution.

4 New Formulations

In Schultz's formulation (Schultz (1992)), we rearrange the variables (and hence, corresponding rows and columns) for the sake of convenience. Without loss of generality, we assume that the first N_1 states, viz., $\{1, 2, ..., N_1\}$ are controlled by player 1 and the rest of the states, viz., $\{N_1 + 1, ..., N\}$ are controlled by player 2. Let N_2= N - N_1.

The LCP is $w - Mz = q = -c$ where $M = \begin{bmatrix} R & B \\ A & 0 \end{bmatrix}$ is a $K \times K$ matrix, K = 4N + (m + n), m = $\sum_{s \in S} m_s$, n = $\sum_{s \in S} n_s$ and the submatrices are defined as follows:

$$R = \begin{bmatrix} 0_{m\times n} & \begin{matrix} -R(1) & 0_{m_1\times n_2} & \dots & 0_{m_1\times n_N} \\ 0_{m_2\times n_1} & -R(1) & \dots & 0_{m_2\times n_N} \\ \vdots & \vdots & & \vdots \\ 0_{m_N\times n_1} & 0_{m_N\times n_2} & \dots & -R(N) \end{matrix} \\ \begin{matrix} R'(1) & 0_{n_1\times m_2} & \dots & 0_{n_1\times m_N} \\ 0_{n_2\times m_1} & R'(2) & \dots & 0_{n_2\times m_N} \\ \vdots & \vdots & & \vdots \\ 0_{n_N\times m_1} & 0_{n_N\times m_2} & \dots & R'(N) \end{matrix} & 0_{n\times m} \end{bmatrix}$$

$B = \left[\, (B_1 - B_2)\ (B_2 - B_1)\ B_3\ -B_3 \,\right]$ where

$$B_1 = \begin{bmatrix} e_{m_1} & 0_{m_1} & 0_{m_1} & \dots & 0_{m_1} & 0_{m_1} & \dots & 0_{m_1} \\ 0_{m_1} & e_{m_2} & 0_{m_2} & \dots & 0_{m_2} & 0_{m_2} & \dots & 0_{m_2} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0_{m_{N_1}} & 0_{m_{N_1}} & 0_{m_{N_1}} & \dots & e_{m_{N_1}} & 0_{m_{N_1}} & \dots & 0_{m_{N_1}} \\ 0_{m_{P_2}} & 0_{m_{P_2}} & 0_{m_{P_2}} & \dots & 0_{m_{P_2}} & 0_{m_{P_2}} & \dots & 0_{m_{P_2}} \\ 0_{n_{P_1}} & 0_{n_{P_1}} & 0_{n_{P_1}} & \dots & 0_{n_{P_1}} & 0_{n_{P_1}} & \dots & 0_{n_{P_1}} \\ 0_{n_{N_1+1}} & 0_{n_{N_1+1}} & 0_{n_{N_1+1}} & \dots & 0_{n_{N_1+1}} & -e_{n_{N_1+1}} & \dots & 0_{n_{N_1+1}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0_{n_N} & 0_{n_N} & 0_{n_N} & \dots & 0_{n_N} & 0_{n_N} & \dots & -e_{n_N} \end{bmatrix}$$

where m_{P_2} is the number of actions for player 1 in player 2 controlled states. That is, $m_{P_2} = \sum_{s_2\in S_2} m_{s_2}$. Similarly, n_{P_1} is the number of actions for player 2 in player 1 controlled states. That is, $n_{P_1} = \sum_{s_1\in S_1} n_{s_1}$.

$$B_2 = \begin{bmatrix} \beta q^1(1,1) & \dots & \beta q^1(1,N_1) & \beta q^1(1,N_1+1) & \dots & \beta q^1(1,N) \\ \beta q^1(2,1) & \dots & \beta q^1(2,N_1) & \beta q^1(2,N_1+1) & \dots & \beta q^1(2,N) \\ \vdots & & \vdots & \vdots & & \vdots \\ \beta q^1(N_1,1) & \dots & \beta q^1(N_1,N_1) & \beta q^1(N_1,N_1+1) & \dots & \beta q^1(N_1,N) \\ 0_{m_{N_1+1}} & \dots & 0_{m_{N_1+1}} & -e_{m_{N_1+1}} & \dots & 0_{m_{N_1+1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0_{m_N} & \dots & 0_{m_N} & 0_{m_N} & \dots & -e_{m_N} \\ e_{n_1} & \dots & 0_{n_1} & 0_{n_1} & \dots & 0_{n_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0_{n_{N_1}} & \dots & e_{n_{N_1}} & 0_{n_{N_1}} & \dots & 0_{n_{N_1}} \\ -\beta q^2(N_1+1,1) & \dots & -\beta q^2(N_1+1,N_1) & -\beta q^2(N_1+1,N_1+1) & \dots & -\beta q^2(N_1+1,N) \\ \vdots & & \vdots & \vdots & & \vdots \\ -\beta q^2(N,1) & \dots & -\beta q^2(N,N_1) & -\beta q^2(N,N_1+1) & \dots & -\beta q^2(N,N) \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0_m & \dots & 0_m & 0_m & \dots & 0_m \\ 0_{m_{N_1+1}} & \dots & 0_{m_{N_1+1}} & -e_{m_{N_1+1}} & \dots & 0_{m_{N_1+1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0_{m_N} & \dots & 0_{m_N} & 0_{m_N} & \dots & -e_{m_N} \\ e_{n_1} & \dots & 0_{n_1} & 0_{n_1} & \dots & 0_{n_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0_{n_{N_1}} & \dots & e_{n_{N_1}} & 0_{n_{N_1}} & \dots & 0_{n_{N_1}} \\ 0_n & \dots & 0_n & 0_n & \dots & 0_n \end{bmatrix}$$

$$A = \begin{bmatrix} -e'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & 0'_{m_N} & e'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ e'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & 0'_{m_N} & -e'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & -e'_{m_N} & 0'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & e'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & e'_{m_N} & 0'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & -e'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & -e'_{m_{N_1+1}} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & e'_{m_{N_1+1}} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & -e'_{m_N} & 0'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & e'_{m_N} & 0'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & 0'_{m_N} & e'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & 0'_{m_N} & -e'_{n_1} & \dots & 0'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & e'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \\ 0'_{m_1} & \dots 0'_{m_{N_1}} & 0'_{m_{N_1+1}} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & -e'_{n_{N_1}} & 0'_{n_{N_1+1}} & \dots & 0'_{n_N} \end{bmatrix}$$

$$c = \begin{bmatrix} 0_{2N+m+n} \\ -e^*_{2N_2} \\ e^*_{2N_1} \end{bmatrix}$$

where e^*_{2k} is the $2k$-dimensional column vector of alternating 1's and -1's, starting with 1.

The following theorem can be easily verified.

Theorem 4.4 *: The matrix M in the above formulation is neither E_0 nor R_0.*

Owing to theorem 2.3 above, we start with the aim of coming up with a formulation that is processible by Lemke's algorithm. But we find that if we artificially try to make the matrix R_0, the E_0 property breaks down and vice versa.

4.1 LCP formulation where we try to achieve the E_0-property:

Among different formulations we tried, following is closest to E_0. This is still not E_0 as can be checked. Artificially trying to impose constraints seems to lead to the breakdown of some condition or the other. Following are the changes we make to the formulation by Schultz (1992).

Here,
$M^1 = \begin{bmatrix} R^1 & B^1 \\ A^1 & 0 \end{bmatrix}$ is a $K_1 \times K_1$ matrix where $K_1 = 3N + (m+n)$.
$R^1 = R$ as in Schultz's formulation; A^1 and B^1 are as follows:

$$A^1 = \begin{bmatrix} e'_{m_1} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & 0'_{n_N} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0'_{m_1} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & e'_{n_N} \\ 0'_{m_1} & \dots & 0'_{m_N} & -e'_{n_1} & \dots & 0'_{n_N} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0'_{m_1} & \dots & 0'_{m_N} & 0'_{n_1} & \dots & -e'_{n_N} \end{bmatrix}$$

$$B^1 = \begin{bmatrix} B_1 & B_3 & B_4 \end{bmatrix}$$

where B_1, B_3 and B_4 are as described in section 3.1 above.

$$c^1 = \begin{bmatrix} 0_{m+n} \\ -e_{2N} \\ e_N \end{bmatrix}$$

4.2 LCP formulation where the underlying matrix is R_0:

We make the following changes to the formulation by Schultz so that M^2 satisfies the R_0 property.
$M^2 = \begin{bmatrix} R^2 & B^2 \\ A^2 & D^2 \end{bmatrix}$ is a $K_1 \times K_1$ matrix where $K_1 = K + 1 = 4N + (m+n) + 1$.
Here too, $R^2 = R$ as in Schultz's formulation, A^2, B^2 and D^2 are as follows:

$$A^2 = \begin{bmatrix} A \\ 0_{m+n} \end{bmatrix}$$

where A is as in Schultz's formulation.

$$B^2 = \begin{bmatrix} B \ 0_{m+n} \end{bmatrix}$$

where B is as in Schultz's formulation.

$$D^2 = \begin{bmatrix} 0_{4N \times 4N+1} \\ \\ -e'_{4N+1} \end{bmatrix}$$

$$c^2 = \begin{bmatrix} c \\ \\ U \end{bmatrix}$$

where c is as described in section 3.1 and U is a large constant.

Notice that we have, in effect, added an additional row and an additional column to the original formulation. This translates to placing a (large) upper bound on the values. The following theorem can be easily checked.

Theorem 4.5 : $M^2 \in R_0$.

5 Conclusion and Future Work

We have provided examples where Lemke's algorithm applied to the LCP formulation as proposed by Schultz (1992) leads to a secondary ray. We propose two new formulations, the underlying matrix in one of which is R_0. It remains open to find an LCP formulation of 2-player, zero-sum, discounted switching control stochastic games that can be processed by Lemke's algorithm. Alternatively, we may formulate the problem as a General (Vertical) LCP and use techniques from (Mohan et al, 1996) to solve them.

Another interesting open problem is to find a suitable d such that our formulations LCP (q, M) find a solution. In fact, this problem is open, in general, for solving LCPs. This problem is of special interest in our case because of the special structure of q because of which we may, in fact, be able to find a suitable d.

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Appendix 1

For the 1-state example discussed in section 3.2.2, the initial tableau is

		z1	z2	z3	z4	z5	z6	w1	w2	w3	w4	w5	w6	z0		q	q/d
w1		0	1	-0.2	0.2	0	0	1	0	0	0	0	0	0		0	-
w2		-1	0	1	-1	-1	1	0	1	0	0	0	0	0		0	-
w3		1	-1	0	0	0	0	0	0	1	0	0	0	0		0	-
w4		-1	1	0	0	0	0	0	0	0	1	0	0	0		0	-
w5		0	-1	0	0	0	0	0	0	0	0	1	0	-1		-1	-1
w6		0	1	0	0	0	0	0	0	0	0	0	1	1		1	-1

Clearly, there is a tie between $w5$ and $w6$ as both correspond to the minimum ratio. Feasibility is maintained irrespective of which of these variables is chosen to leave the basis. If we choose $w5$, then we get the following tableau.

		z1	z2	z3	z4	z5	z6	w1	w2	w3	w4	w5	w6	z0		q	q/z5
w1		0	1	-0.2	0.2	0	0	1	0	0	0	0	0	0		0	-
w2		-1	0	1	-1	-1	1	0	1	0	0	0	0	0		0	-
w3		1	-1	0	0	0	0	0	0	1	0	0	0	0		0	-
w4		-1	1	0	0	0	0	0	0	0	1	0	0	0		0	-
z0		0	1	0	0	0	0	0	0	0	0	-1	0	1		1	-
w6		0	0	0	0	0	0	0	0	0	0	0	1	0		0	-

As $w5$ has left the basis, the entering variable is $z5$. But as none of the entries in the column corresponding to $z5$ are positive, we have reached a secondary ray and Lemke's algorithm cannot proceed further. Note that this game is also a one-player control game and Lemke's algorithm is not guaranteed to terminate for this formulation even for the one-player control case.

For this example, it is easy to choose d so as to force $w6$ to correspond to the row with minimum ratio in the first iteration. For example, choose $d = \begin{bmatrix} 0 \ 0 \ 0 \ 0 \ 2 \ -1 \end{bmatrix}'$. On choosing $w6$ as the leaving variable, it is easy to verify that all paths lead to a solution.