Modeling of Extreme Market Risk using Extreme Value Theory (EVT): An Empirical Exposition for Indian and Global Stock Indices

Gaurav Singh Chauhan

ABSTRACT

The paper presents an empirical application of Extreme Value Theory (EVT) in modeling extreme behavior of financial asset prices. EVT is a branch of statistics dealing with the extreme deviations from the mean of probability distributions. It is the theory behind modeling the maxima of a random variable. Market risk takes extreme form when certain events, which are assumed to be rare in the distribution of assets, cause severe changes in the valuation of the portfolio. These rare events usually lie in the tails of the return distribution of the assets. We model two different risk measures with three different return distributions and analyze the importance of extreme value distributions in underscoring the behavior of extreme market moves. The paper highlights that the high volatility of Indian stock market associated with higher returns is better captured through extreme value distributions rather than conventional distributions. Also, for all the stock indices, EVT help in modeling the market risk better than conventional distributions.

Key Words: Market risk, Extreme Value Theory (EVT), enhanced risk measures, extreme value distributions

JEL Classification: C12, C13, C15, C19, G11

1. Introduction

Research in recent times, especially in the latter half of 20th century, dealt with the determination of an explicit trade-off between risk and returns. While there is almost no disagreement on the definition and specification about returns ascribed to an asset, researchers do disagree with the notion of risk. While the specific definition of risk is very important when used as the stochastic discount factors for asset pricing, it is equally important to estimate an aggregate measure of risk in portfolio of asset for determination of risk capital.

Market risk is the risk faced by a portfolio of assets due to change in market moves or market wide risk factors. Market risk takes extreme form when certain events, which are assumed to be rare in the distribution of assets, cause severe changes in the valuation of the portfolio. These rare events usually lie on the tails of the return distribution of the assets. These rare events may therefore, cause a large gain or a large loss to the value of the portfolio. As an average investor is concerned with the extreme losses rather than large gains, we are also concerned with the loss severity or the downside risk of the portfolio.

The conventional analysis of risk return trade-off starts with the objective quantification of risk by Markowitz (1952). This was the first instance to determine a mathematical formulation of risk-return trade-off in the form of efficient frontier. In his analysis, Markowitz was able to demonstrate the diversification of asset portfolio through multivariate distribution of returns using Pearson's correlation coefficient. While the work was inherently seminal in its own context, recent extreme events in the market led us to relook the assumptions underlying the Markowitz framework. The analysis relied on elliptically distributed asset classes, where Pearson's correlation is a well behaved concordance metric [1]. The elliptically distributed random variable is one for which the density can always be presented as a nondegenerate variance mixture of (at least two) normal densities (Owen and Rabinovitch, 1983). If we happen to look into the return distributions for most of the assets in extreme market moves, we may find strong disagreement with the assumption of elliptical distributions. These distributions tend to show skewed and leptokurtic patterns. Under these conditions a Pearson's correlation of zero does not signify the independence of asset returns.

The above background laid the foundation and motivation for this work. Emerging markets such as India entails higher volatility associated with higher returns. Capturing market risk in these markets effectively would therefore have larger implications for global investors to diversify their risk through quantifiable exposures in these markets. In this paper we analyze risk in more than one dimension using VaR and Expected Shortfall as alternative risk measures. We then applied these risk measures to different kinds of asset distributions to see the utility and fit of these measures for different asset classes in the form of real time financial time series of Indian and global stock indices. The exercise thus helps us to compare the performance of various risk measures and return distributions with respect to chosen stock indices.

Section 2 gives a theoretical background regarding definitions of various risk measures and concept of extreme value distributions. These aspects necessarily deal with properties of risk measures, Extreme Value Theory (EVT) and its use in extreme distributions. Although, the theoretical description of risk measures and EVT would be dealt with in details in advanced texts on Risk and their properties, this is included here to make definitions of these measures consistent as used for the empirical analysis in later sections. An avid practitioner can skip this section entirely. In section 3 we reviewed some key works related to the paper. Section 4 reveals the data and methodology for the empirical analysis. In this section we intend to show how modeling of Value-at-Risk (VaR) and Expected Shortfall (ES), as alternative risk measures to the variance, are modeled for a set of asset returns consisting of actual financial time series using three different return distributions. Section 5 discusses the results and implications. Section 6 provides the concluding remarks with some limitations.

2. Theoretical Background

2.1. Extreme Value Theory (EVT)- Generalized Extreme Value (GEV) distributions and Generalized Pareto Distributions (GPD)

EVT is the theory behind modeling the maxima of a random variable. It is a branch of statistics dealing with the extreme deviations from the mean of probability distributions. Extreme value theory is important for assessing risk for highly unusual events. Let X_n be the realization for any random process W in period n, with specified maximas for any set of non-overlapping periods of fixed intervals. We therefore, are interested in modeling the distributions of $M_n = \max\{X_1, X_2 \ldots X_n\}$. This can be done by observing the maxima in two ways. First is the Block Maxima Method and second is the Peak Over Threshold method (POT). Under the former, we consider the maximum value of variables in successive non-overlapping periods and determine the distributions of these maximas. In the latter, however, we specify a threshold and define maxima as the value of random variable exceeding this threshold. The distribution of these maximas follows Generalized Extreme Value (GEV) distributions. Using the fundamental theorem of Fisher & Tipper (1928) and Gnedenko (1943), Jenkinson (1955) has been cited to show the formulation of GEV distributions. The limiting case of maxima, $M_n = F(\mu, \sigma, \xi)$ is given by:

$$F(x;\mu,\sigma,\xi) = \left\{ -\left[1 + \xi\left(\frac{x-\mu}{\sigma} \right) \right]^{-1/\xi} \right\} \quad \ldots\ldots (1)$$

for $1 + \xi(x - \mu) / \sigma > 0$, where $\mu \in R$ is the location parameter, $\sigma > 0$ the scale parameter and $\xi \in R$, the shape parameter.

Fisher-Tipper type I, II and III distributions are said to be obtained respectively for $\xi = 0$, $\xi < 0$, and $\xi > 0$. Depending on the range of the parameters these distributions are called Frechet, Weibull and Gumbel distributions. Maximum domain of attraction (MDA) is the concept from seminal work of Fisher & Tipper,(1928). McNeil (1996) discussed MDA in detail. Under the concept of MDA, if we know that suitably normalized maxima converge in distribution, then the limit distribution must be an extreme value distribution for some value of the parameters μ, σ, and ξ. Using MDA different sub-class of distributions are classified, such as Pareto, Burr, normal, exponential etc.

Under POT method of estimating the maxima a threshold value, u, has been considered and the excess distribution function F_u of the values of random variable exceeding u has to be obtained. For an asymptotic case where $u \to \infty$, under conditions of MDA, F_u assumes the form of Generalized Pareto Distributions (GPD), which is expressed as (Pickands, 1975; Balkema & de Hann, 1974):

$$F(x;\mu,\sigma,\xi) = \left\{1 - \left[1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\} \quad \text{.......................... (2)}$$

For $x > \mu$, and $x \le \mu - \sigma/\xi$ when $\xi < 0$, where μ is the location parameter, σ the scale parameter and ξ the shape parameter.

The tail index, ξ, reflects the heaviness of tails while the scale, σ parameter reflects the dispersion of the distribution around the mean. The higher the ξ, heavier is the tail. Different values of these parameters results in different forms of GPD like Pareto ($\xi > 0$), Uniform ($\xi = 1$), Exponential ($\xi = 0$). To estimate these parameters two general methodologies used are (i) semi-parametric method based on Hill type estimators (Hill 1975) and, (ii) Parametric methods based on specific GPDs to be used. Danielsson and de Vries (1997) use semi-parametric approach based on Hill estimator for estimating the tail index.

Hosking & Wallis (1987) gave a detailed analysis on the parameter estimation of GPD. For given random sample $X_1, X_2 \ldots X_n$ of a population X from the GPD, the Maximum Likelihood Estimators (MLEs) of the parameters (σ, ξ) have been discussed by them. The Moment Estimators (MEs) of the parameters are given by:

$$\hat{\sigma}_{ME} = \overline{X}\left(\frac{\overline{X}^2}{s^2} + 1\right)/2, \text{ and } \hat{\xi}_{ME} = \overline{X}\left(\frac{\overline{X}^2}{s^2} - 1\right)/2 \quad \text{.......... (3)}$$

where $\overline{X}$ and s^2 are the sample mean and the sample variance respectively.

The probability-weighted moment (PWM) estimators of the parameters for the GPD:

$$\hat{\sigma}_{PWM} = \frac{2a\overline{X}}{\left(\overline{X} - 2\alpha\right)} \text{ and } \hat{\xi}_{PWM} = \frac{X}{\left(\overline{X} - 2\alpha\right)} - 2 \quad \text{................ (4)}$$

where, $\alpha = n^{-1}\sum_{i=1}^{n}\frac{n-i}{n-1}$ with X_i being the ith order statistic.

The traditional methods of estimating parameters for GPD are found to be difficult to estimate. Since Maximum Likelihood estimation (MLE) is computationally difficult, approximations given by ME estimators are often used. Davison (1984), Smith (1985), Hosking & Wallis (1987) and Grimshaw (1993) discussed the problems in these estimations.

2.2. Risk measures and their properties

Many applications in quantitative finance need the risk in a portfolio of assets to be quantified. This quantification generally is expressed as risk exposure. Among the set of models to estimate the risk exposure, Value-at-risk (VaR) is the one which is widely used. VaR is mathematically expressed as:

$VaR_\alpha = \inf\{l \in R : p(L>l) \le 1 - \alpha\} =$
$\inf\{l \in F_L(l) \ge \alpha\}$ .... (5)

This can be interpeted for a confidence level α, the smallest number *l* such that the probability that the loss *L* exceeds *l* is not larger than $(1 - \alpha)$. Or VaR as the lowest value of the random variable X, denoting the portfolio returns, that can be achieved with a confidence α. In other words VaR is the α-quantile of the generalised inverse of distribution F(X) of the portfolio returns. VaR has been used extensively in financial applications due to its intuitive appeal, easy implementation and straightforward applicability.

Three general methods to calculate VaR are (i) empirical estimation on historical returns, (ii) using parametric estimation of distribution functions and (iii) Monte carlo simulations. Some specific VaR calculations techniques are :

1. Delta-normal model (Garbade, 1986),

2. Delta GARCH model (Hseih, 1993),

3. Delta weighted normal model (Morgan, 1994),

4. Gamma normal model (Wilson, 1994).

Researchers have cited several problems in using VaR as a risk metric. VaR is not a coherent risk measure (not even a weak coherent measure). Thus it is difficult to get single estimate for VaR using different methods for the same portfolio returns. VaR lacks the sub-additivity property (discussed below), where it goes against the tenets of portfolio diversification. It is not a convex function of risk, which causes computational issues in optimization problems, when used as a constraint. Winker & Maringer (2004) have attempted to optimize a portfolio using VaR through the use of mathematically sophisticated Memetic search algorithms.

VaR seems to penalize gains and losses in the same way and is also inappropriate for rare events. It is also inconsistent with the utility function approach except when assets have Gaussian distribution or when quadratic utility is assumed. Thus it is unsuitable for modeling risk in case of non-elliptical distributions.

2.2.1. Coherence of risk

To see the coherence of risk measures, let ρ be the risk measure, defined as the amount invested prudently today to bear an expected loss of X in future. Artzner *et al.* (1999), defined the following axioms for a coherent measure of risk given the mapping ($\rho : X \rightarrow R$) :

a. Translation invariance, i.e., $\rho(x + r) = \rho(x) - a$ for all random variables x, real number 'a' and riskless rate r_f;

b. Sub-additivity, i.e., $\rho(x + y) \leq \rho(x) + \rho(y)$ for all random variables x and y;

c. Positive homogeneity, i.e., $\rho(\lambda x) = \rho(\lambda x)$ for all random variables x and positive real numbers λ and,

d. Monotonicity, i.e., $x \leq y \Rightarrow \rho(x) \leq \rho(y)$ for all random variables x and y.

It can be noted again that VaR does not obey the sub-additivity property.

2.2.2. Different risk measures

a. Conditional value at risk (CVaR)

CVaR is defined as the expected value of losses exceeding VaR. Mathematically,

$$CVaR(X) = VaR(X) + E[X - VaR(X) \mid X > VaR(X)] \ldots(6)$$

CVaR is a coherent measure having all the properties of a coherent risk measure.

b. Expected Shortfall (ES)

For an integrable loss distribution F(X) satisfying,

$$\int_R |x|.df(x) < \infty$$

ES is defined as the conditional expectation of the random variable X, such that X > VaR X. Mathematically, ES (X) =

$$E[X \mid X \geq VaR(X) + E[X - VaR(X) \mid X > VaR(X) \ldots..(7)$$

The condition of sub-additivity holds for ES and it is a coherent risk measure.

c. Expected regret (ER)

ER is very similar to CVaR, where it is defined as the expected value of loss distribution given it exceeds a threshold, β. Mathematically,

$$ER_\beta(X) = \int f(x.y) - \beta .p(y).dy \ldots\ldots\ldots\ldots\ldots(8)$$

Where, p(y) is the probability density function of the loss distribution.

d. *Tail mean (TM)*

TM is defined as:

$$TM_\alpha(x) = \frac{1}{\alpha}\left\{E\, XI_{x \leq x(\alpha)} + x_\alpha\ \alpha - p(X \leq x(x)\right\} \ldots\ldots.(9)$$

Where, $I_{X \leq x(\alpha)}$ is the indicator function defined such that $I_{X \leq x(\alpha)} = 1$ if $X \leq x(\alpha)$ and 0 otherwise and α is the confidence level.

3. Literature Review

3.1. Extreme Value Theory (EVT) - Generalized Extreme Value (GEV) distributions and Generalized Pareto Distributions (GPD)

The evidence as given by Fama (1976) generally suggests that the distribution of daily returns is heavy-tailed distributions relative to normal. Mandelbrot (1966) was among the first ones to acknowledge the heavy tails and excess peakedness underlying the real time financial assets. The concept in fact was earlier proposed by Jenkinson (1955) in terms of introducing Generalized Extreme Value (GEV) distributions. This led to the path breaking venture into the field of Extreme Value Theory (EVT). Researchers since 1990s were constantly seeking answers to some real time risk phenomenon using the realm of EVT. EVT in recent times proved to be instrumental in statistically modeling rare events which are of considerable importance for aggregation of risk in a portfolio. EVT has been extensively used nowadays to compute point estimates and confidence intervals for tail risk measures in a financial optimization problem. Embrechts et al. (1997) showed that EVT specially focuses on behavior of tail dependence for set of asset returns and used for modeling the maxima of a random variable. Other distributions such as Stable Paretian distributions are also modeled to deal with heavy tails; here these distributions, however, deal with complete distributions and not only the tails. A wide description of Stable Paretian distributions for financial assets has been given in Rachev and Mittnik,(2000). Beirlant & Teugels (1992), Embrechts & Kluppelberg (1993), gave a mathematical treatment about insurance mathematics in EVT and its applications.

On the application front EVT finds wide utility. Neftci (2000) compared VaR based on Normal and extreme value distribution assumptions using historical bond prices and foreign exchange rates. Da Silva and Vaz de Melo Mendez (2003) computed VaR estimates using EVT to analyze ten Asian stock markets. Gilli and Këllezi (2003) also advocate EVT, Block Maxima and Peak Over Threshold (POT) to compute tail risk measures: VaR and ES. These methods to identify the maxima for GEV distributions are discussed later in the paper. Embrechts et al. (1997) modeled rare events in insurance and other quantitative finance aspects using EVT. Longin (2000) in his paper showed the implementation of EVT for estimating VaR of a portfolio. Extending the concept of EVT to the insurance industry, McNeil (1996) used the Danish insurance data to highlight the relevance of Generalized Pareto Distributions (GPD), which is a sub-class of GEV, for EVT. He also dealt with the parameter estimation and curve fitting for modeling rare historical losses in non-insurance sector. He also dealt with the concept of loss severity and showed how to model the aggregate payments depending on the number of losses. He employed the method of Maximum Likelihood Estimation (MLE) as well as Probability Weighted Method (PWM) of moments for parameter estimation and data fitting and found that GPD is the best fit distribution for extreme values.

A crucial aspect in modeling with GPD is the parameter estimation and curve fitting. Jenkinson (1955) and Prescott & Walden (1980), dealt with the estimation of GEV parameters. They used Maximum Likelihood Estimation (MLE) and sextile estimation methods for estimating parameters for GEV distributions. There are three functional forms for estimating GPD parameters. Maximum Likelihood Estimators (MLE), Probability Weighted Method (PWM) and moment estimators (ME). The Maximum Likelihood Estimators (MLEs) of the parameters have been discussed by Davison (1984), Smith (1985), Hosking & Wallis (1987) and Zhang(2007). Grimshaw (1993) gave a detailed algorithm for computing the MLEs for some restrained set. Hosking & Wallis (1987) also discussed an approximation in terms of the moment estimators (MEs) of the parameters. They also discussed the PWM estimation in the same paper.

3.2. Risk measures

Disagreement among researchers for appropriate risk measure in portfolio optimization led to proper definition of essential properties of risk measures. Artzner et al. (1999) in his seminal work has worked for defining the axioms of a coherent risk measure. VaR generally does not possess the sub-additivity property, which goes against the tenet of portfolio optimization. Different risk measure and their properties have been dealt with by some researchers[1]. Zokivic (2008) dealt with CVaR and properties of other risk measures. In emerging market context, Darbha (2001) investigated the value-at-risk for fixed income portfolios in India, and compared alternative models including variance-covariance method, historical simulation method and extreme value method without making distribution assumptions for entire return processes. Also Lima and Neri (2007) investigated the VaR methodologies for Brazillian market.

The problem of optimization of a portfolio with different risk metrics for elliptically distributed assets is dealt in Giorgi (2002). He showed that mean-VaR and mean-ES optimization are subsets of mean-variance efficient frontier. Rockafellar & Uryasev (2002) discussed this comparison for multivariate Gaussian distributions. The formal definition and properties of the two risk measures have been discussed in section 2 above.

4. Data and Methodology

In the following empirical analysis we have modeled different return distribution against different risk measures such as VAR and ES. We chose historical estimation (empirical distribution), Gaussian distribution and Generalized Pareto Distribution using Peak Over Threshold (POT) method to estimate VAR and ES of real time financial asset returns. Our data set consists of financial time series of four stock market indices viz SENSEX, CNX NIFTY, S&P 500 and FTSE 100.

We chose daily returns for these time series for the period starting from January 1995 to December 2009. The period was chosen such that to capture broad market movements after structural changes have been introduced in Indian economy in early 1990s. Further the tail of the data set captured the period of extreme market movements owing to recent global meltdown of 2007-09. In all we have sample points data for 3692 observations during this period.

1 See, for example, Duffie & Pan (1997), Gordy (2000), Jackson & Perraudin (2000), Jorion (1996), Rockafellar & Uryasev (2002), Szego (2005) etc .

We estimated the VAR and ES for the different return distribution sequentially starting from empirical distribution. Microsoft Excel and SPSS 16.0 were used for analyzing the data.

a. Empirical distribution:

Let F_n denote the empirical process of the observed losses $X_1, \ldots, X_n$,.

This is given by

$$F_n(t) = \frac{l}{n}\sum_{i-1}^{n} l(X_i \geq t) \text{........}(10)$$

where $l(\cdot)$ is the indicator function, and X_i is i.i.d. with (unknown) distribution F.

Let $X_{n(1)} \leq X_{n(2)} \leq \ldots \leq X_{n(n)}$ is the order statistics, then the VaRα(X) or α quantile $F^{-1}(\alpha)$ can be estimated by (Van Der Vaart, 1998) :

$$F_n^{-1}(\alpha) = X_{n(i)}, \alpha \in \left(\frac{i-1}{n}, \frac{i}{n}\right) \text{............}(11)$$

In effect we calculate the α percentile of the return distribution for given data points.

Similarly, ES is given by,

$$ES = E(x \mid x > VAR(X)) \text{ ....}(12)$$

In effect, assuming i.i.d returns ES is the average of loss data points exceeding VAR.

b. Gaussian Distribution

Let X_i, $i=1,\ldots,n$, be i.i.d., with normal distribution $X \sim N(\mu, \sigma^2)$ with unknown μ and σ. Then VaR at α confidence level is simply given by z_α σ, where z_α is such that $P(Z > z_\alpha) = \alpha$, with $Z \sim N(0,1)$. Where σ, the standard deviation can completely define VAR for a given confidence level. This means that for a given confidence, say 95%, the VaR is simply 1.645σ and for 99% CI VaR is 2.33σ.

Similarly ES is given by Eq (12) above,

Where, $z_\alpha = \Phi^{-1}(\alpha)$ represents the α quantile of the standard Normal distribution, Φ is the cumulative distribution function (cdf) of the standard Normal distribution. ES can be approximated once the value of VAR is known. For 95% CI, the ES is approximately 2.06σ and for 99% CI, ES= 2.67σ. We have used these approximations here.

c. EVT approach - GPD distribution

Earlier we have seen that to estimate maxima of a random variable we can use either the block maxima method (BMM) or the peak over threshold (POT) method. BMM uses the value of maxima in successive non-overlapping intervals, while a threshold is to be defined in case of POT. The random variable which is greater than the threshold is under consideration. The distribution function of these maxima constitutes the GPD which is given above in Eq (2). For such a GPD, VAR and ES are given by, (Lee, 2009).

$$VaR = u + \frac{\sigma}{\xi}\left(\left(\frac{n}{n_u}(1-q)\right)^{-\xi} - 1\right) \qquad (13)$$

$$ES_q = \frac{VaR_q + (\sigma - \xi\mu)}{1-\xi} \qquad (14)$$

Where, p = confidence level=1-q; σ = scale parameter; μ = mean, u =threshold ξ = shape parameter, or tail index, n_u is the number of data sets exceeding the threshold u and n is the total data points. Moment estimators in equation 3 can be used here as approximations to estimate σ and ξ.

We used a daily forecast of VAR and ES based on rolling window approach of Harmantzis et al (2006). The rolling period considered is 125, 250 and 500 days. Thus for 126th day we estimate VAR and ES using data from day 1 to day 125 for all the three distributions mentioned above. Similarly 127th day forecast is generated by rolling the window one day ahead i.e day 2 to day 126 etc. Similar approach is followed for other windows of 250 and 500 days. We have used 95% as the confidence level.

VaR and ES for these windows of time period is calculated for all the three distributions mentioned above using the equations underlying these distributions for VaR and ES. Under POT method for GPD the threshold is set at 50% of maxima in the time window. This is done because there is no standard methodology to choose a threshold in POT method (Embrechts et al., 1997) and our objective was to demonstrate the utility of threshold method rather than achieving definite results.

We also performed measure-of-fit test for VaR and ES for all three distributions. For VaR we record the violations, which were the returns in the data set exceeding the forecasted VaR. If the model was a perfect fit then, VaR violations must be $1-\alpha$, where α was the confidence level. These violations were statistically tested using binomial testing. If the violations were statistically significantly less than $1-\alpha$, then the model was overstating VAR and vice versa. Thus null hypotheses that the model was a good fit could be tested using binomial tests. The null can be rejected if the p-value is less than 0.05.

For ES, it is to be noted that when VaR violations occur, the difference between actual returns and ES must be statistically zero for a good fit model. This difference can be statistically tested using t-test with the null hypotheses of the difference being zero. We can reject the null hypotheses if the p-values are less than 0.05.

5. Results

First the VaR calculations were done for all the time series and for three different kinds of distributions. The VaR violations, defined as data points where daily return exceeds that of calculated VaR, was noted and the total number of such violations were compared to the expected violations. The number of expected violations must be 5% for 95% confidence level. If the violation exceeds the 5% level then VAR was underestimating the risk and vice versa. To check for the difference between VAR violations and that which were expected, Binomial test was employed. The results for different time series and for different window sizes were tabulated below in Table 1.

From Table 1 it can be seen that the Empirical distribution significantly overestimate VaR for all window sizes. For Gaussian distribution, except FTSE 100, VaR model properly fitted the data for other time series. Window size of 125 seems to be a proper fit for valuing VaR through Gaussian distribution and 500 size window significantly overestimates the risk. Similar results are obtained for EVT (GPD) distribution. Except FTSE, VaR model properly fits for other time series. Window size

Table 1 : Binomial Test for VaR violations

	SENSEX			NIFTY			S&P 500			FTSE 100		
Empirical Distribution												
Expected	Series	Actual	Siqnifi.	Series	Actual	Siqnifi.	Series	Actual	Siqnifi.	Series	Actual	Siqnifi.
178.3	BSEH125	210	0.0095	NSEH125	219	0.0014	SPH125	215	0.0034	FTH125	219	0.0014
172.05	BSEH250	202	0.0120	NSEH250	204	0.0081	SPH250	208	0.0035	FTH250	208	0.0035
159.55	BSEH500	191	0.0071	NSEH500	186	0.0193	SPH500	214	0.0000	FTH500	215	0.0000
Gaussian Distribution												
178.3	BSEG125	190	0.1939	NSEG125	191	0.1739	SPG125	193	0.1380	FTG125	212	0.0063
172.05	BSEG250	178	0.3315	NSEG250	173	0.4813	SPG250	199	0.021 0	FTG250	205	0.0066
159.55	BSEG500	175	0.1133	NSEG500	164	0.3700	SPG500	204	0.0003	FTG500	212	0.0000
EVT (GPD) Distribution												
178.3	BSEE125	188	0.2380	NSEE125	191	0.1739	SPE125	193	0.1380	FTE125	211	0.0078
172.05	BSEE250	178	0.3315	NSEE250	173	0.4813	SPE250	198	0.0251	FTE250	205	0.0066
159.55	BSEE500	174	0.1292	NSEE500	163	0.4009	SPE500	204	0.0003	FTE500	212	0.0000

of 125 seems appropriate here also and 500 days window is least preferred.

We also conducted ES calculations for all the time series and all the size windows. For estimating the fit for the models we looked for the points where VaR violations occurred and estimated the difference between actual returns and estimated ES. It can be noted here that this difference for a better model must be statistically zero. Thus we applied the t-test to measure whether the mean difference between actual returns and the ES is different from zero. The results for t-test are tabulated below in Table 2.

It can be seen from Table 2 that for Empirical distribution, window size of 125 days does not properly relate to the data and ES consistently underestimates the risk. This fit is improved for all the time series, except FTSE 100, for window sizes of 250 and 500 days. Gaussian distribution seems to be barely fitting the SENSEX and NIFTY. For these time series ES consistently underestimates the risk. For window size of 500 ES shows same pattern for FTSE 100 too. Rest of the time series window combinations seems appropriately

Table 2: t-test to measure the mean difference between actual returns and the ES

Empirical Distribution															
	SENSEX				NIFTY				S&P500				FTSE100		
Series	N	Mean	Signif.	Series	N	Mean	Signif.	Series	N	Mean	Signif.	Series	N	Mean	Signif.
BSEH125	210	-0.0031	0.0007	NSEH125	219	-0.0026	0.0060	SPH125	215	-0.0019	0.0028	FTH125	219	-0.0014	0.0135
BSEH250	202	-0.0018	0.0724	NSEH250	204	-0.0013	0.1926	SPH250	208	-0.0013	0.0676	FTH250	208	-0.0017	0.0044
BSEH500	191	-0.0015	0.1566	NSEH500	186	-0.0019	0.0957	SPH500	214	-0.0015	0.0632	FTH500	215	-0.0020	0.0025
Gaussian Distribution															
BSEG125	190	-0.0020	0.0434	NSEG125	191	-0.0024	0.0219	SPG125	193	-0.0008	0.2745	FTG125	212	0.0001	0.9256
BSEG250	178	-0.0024	0.0201	NSEG250	173	-0.0030	0.0089	SPG250	199	-0.0012	0.1310	FTG250	205	-0.0011	0.0865
BSEG500	175	-0.0022	0.0508	NSEG500	164	-0.0039	0.0015	SPG500	204	-0.0024	0.0034	FTG500	212	-0.0025	0.0004
EVT(GPD)Distribution															
BSEE125	188	-0.0011	0.2376	NSEE125	191	-0.0011	0.2613	SPE125	193	-0.0007	0.3452	FTE125	211	0.0001	0.8414
BSEE250	178	-0.0016	0.1305	NSEE250	173	-0.0019	0.0869	SPE250	198	-0.0009	0.2368	FTE250	205	-0.0008	0.1800
BSEE500	174	-0.0021	0.0645	NSEE500	163	-0.0032	0.0097	SPE500	204	-0.0019	0.0231	FTE500	212	-0.0020	0.0028

relating to the data. GPD seems to be a better model than both the Empirical distribution and the Gaussian distribution. Except for window size of 500 days GPD shows a good fit for all the time series and for all the window sizes.

6. Managerial Implications

As has been witnessed in recent market moves, financial markets are increasingly being exposed to both exogenous and endogenous shocks. An important feat in managing these uncertainties is to precisely predict these shocks and design models for their remedy. Rather than investing in resources to estimate the timings of such volatilities, it seems to be a much better option to estimate the appropriate risk exposure in dealing with particular financial assets. Following such an approach risk measures such as VaR and ES comes handy in estimating these exposures.

However, appropriate fit between these risk measures to the concerned distribution of asset prices would be quite challenging for risk managers. Ex post estimation of parameters of well known distribution may not serve the purpose because of inherent lack of fit of these distributions to real time series data.

Managers are therefore left with relative decision making where while working with different distributions they tend to attain certain confidence in fitting these distributions to real time series data. Empirical analysis such as the one dealt with in this paper helps us to make use of certain distributions which represents the volatility in financial time series pretty well. However, caution needs to be applied in such uses of these distributions with certain risk measures as some distributions might look to be artificially suited to these risk measures.

7. Conclusion and Discussion

We have studied EVT and its features with the help of GPDs which are hypothesized to describe the risk of rare events in a better manner. Shortfall of VaR as a risk led us to search for better options; that can capture rare events with more certainty. To empirically interpret the relation between different distributions and different risk measures, we empirically tested the fit for two of the risk measures viz VaR and ES to three different distributions namely empirical, Gaussian and EVT (GPD) for four real time series consisting of Indian and global stock indices. We relied on previous studies to lead us through the introduction of EVT and different risk measures and to guide us for empirical testing.

The findings point towards the relative importance of EVT in exploring dynamics of rare events in a distressed market setting. Higher volatility of Indian stock indices which are associated with their higher returns are seen to be better encapsulated by extreme value distributions, which enriches risk measures. Also, for all the stock indices under consideration, market risk seems to be better modeled using EVT rather than conventional risk measurements.

The empirical analysis made use of approximations suggested in previous research for deriving moment estimators. The accuracy and mathematical convergence of these can be improved significantly with sophisticated mathematical softwares and algorithms. Parameter estimation for GPD is one such challenging area that needs to be further explored with respect to mathematical efficiency and relative utility for approximations have to be made. Moreover, the analysis can be better reinforced by further tests.

Although we have primarily checked for significant serial correlations in time series that have been used, some of these tests may take better care of associated serial correlations; they may be used in checking clustering of extreme market moves in certain period of analysis. This area provided us a ground for further research in refining the finding and draw seasoned inferences thereof.

Acknowledgement

The author is thankful to Prof. L.V. Ramana, (Faculty, (F&A), IIM Indore) for his valuable inputs and suggestions for some of the ideas developed in the paper.

Notes

1. Concordance is the basic measure of association between two random variables. Two pairs of observation on continuous random variables X and Y denoted by (x_1, y_1) and (x_2,y_2) are said to be concordant if x_1-x_2 is of the same sign as y_1-y_2. That is the pairs are concordant if:

   $(x_1-x_2)(y_1-y_2)>0$ and discordant if $(x_1-x_2)(y_1-y_2)<0$.

Following are the properties of a concordant metric m(X,Y):

1. m(X,Y) ? [-1,1],
2. m(X,Y) = 0, if X and Y are independent
3. if F(X,Y) G(X,Y) then, mF(X,Y) mG(X,Y); where F(.) and G(.) are two possible joint distributions.

References:

Artzner, P., Delbean. F., Eber J.-M.,and Heath, D., (1999). Coherent measures of risk. *Mathematical Finance,* 9, 203-228.

Balkema, A. A., and de Hann, L., (1974). Residual life time at great age. *Annals of Probability,* 2, 792-804.

Beirlant, J., and Teugels, J., (1992). Modelling large claims in non-life insurance. *Insurance: Mathematics and Economics,* 11, 17-29.

Da Silva, A.L.C., and Vaz de Melo Mendes, B. (2003), Value-at-risk and extreme returns in Asian stock markets. *International Journal of Business,* 8, 17-40.

Danielsson, J., and de Vries, C. (1997). Tail index and quantile estimation with very high frequency data. *Journal of Empirical Finance,* 4, 241-57.

Darbha,G. (2001). Value-at-Risk for Fixed Income portfolios: A comparison of alternative models. *National Stock Exchange, Mumbai,* India.

Davison, A.C. (1984). Modeling excesses over high thresholds, with an application. In Statistical Extremes and Applications, ed. J. Tiago De Oliveira, pp. 461-482, Dordrecht: Reidel.

Duffie, D., and Pan, J., (1997). An overview of value at risk. Journal of Derivatives, 4, 7-49. Reprinted in Options Markets, (2000), G. Constantinides and A. G. Malliaris (edited), *Edward Elgar, London, UK.*

Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). Modelling Extremal Events for Insurance and Finance. *Springer Verlag,* Berlin.

Embrechts, P., and Klueppelberg, C., (1993). Some aspects of insurance mathematics. *Theory of Probability and Applications,* 38, 262-295.

Fama, E.F. (1976). Foundation of Finance. Basic Books, New York

Fisher, R., and Tippett, L. H. C., (1928). Limiting forms of the frequency distribution of largest or smallest member of a sample. *Proceedings of the Cambridge Philosophical Society,* 24, 180- 190.

Garbade, K., (1986). Assessing risk and capital adequacy for treasury securities. *Topics in money and security markets, Bankers Trust.*

Gilli, M., Këllezi, E. (2003). An application of extreme value theory for measuring risk. Working Paper. Department of Econometrics, University of Geneva and FAME, Geneva.

Giorgi, E., (2002). A note on portfolio selection under various risk measures. *Swiss Federal Institute of Technology Zurich, department of Mathematics.*

Gnedenko, B. V., (1943). Sur la distribution limited u terme d'une serie aleatoire. *Annals of Mathematics,* 44, 423-453.

Gordy, M. B., (2000). A comparative anatomy of credit risk models, *Journal of Banking and Finance,* 24, 119-149.

Grimshaw, S.D. (1993). Computing maximum likelihood estimates for the generalized Pareto distribution. Technometrics 35, 185-191.

Harmantzis, F. C., Miao, L., and Chein, Y., (2006). Empirical study of value-at-risk and expected shortfall models with heavy tails. *The Journal of Risk Finance,* 7,117-135

Hill, B. M., (1975). A simple general approach to inference about the tail of a distribution. *The Annals of Statistics,* 3, 1163-1174.

Hosking, J. R. M., Wallis, J. R., (1987). Parameter and quantile estimation for the generalized Pareto distribution. Technometrics 29, 339-349.

Hseih, D., (1993). Implications of non-linear dynamics for financial risk management. *Journal of Financial and Quantitative Analysis,* 28, 41-64.

Jackson, P., and Perraudin, W., (edited), (2000). Special issue on credit risk modeling and regulatory issues. *Journal of Banking and Finance,* 24.

Jenkinson, A. F., (1955). The frequency distribution of the annual maximum (minimum) values of meteorological events. *Quarterly Journal of the Royal Meteorological Society,* 81, 158-172.

Jorion, P., (1996). Risk: Measuring the risk in value at risk. *Financial Analyst Journal,* 52, 47- 57.

Lee, W. C., (2009). Applying Generalized Pareto Distribution to the Risk Management of Commerce Fire Insurance. *Preliminary draft, Department of Banking and Finance Tamkang University,* Taiwan

Lima,L.R. and Neri,B.P.(2007). Comparing Value-at-Risk Methodologies. *Brazilian Review of Econometrics,* 27, 1-25

Longin, F.M. (2000). From value at risk to stress testing: the extreme value approach. *Journal of Banking and Finance,* 24, 1097-130.

Mandelbrot, B., (1966). Forecasts of Future Prices, Unbiased Markets, and 'Martingale' Models. *Journal of Business,* 39, 242-255.

Markowitz, H. M., (1952). Portfolio selection. *Journal of Finance,* 7, 77-91.

McNeil, A. J., (1996). Estimating the tails of loss severity distributions using extreme value theory. *Swiss Federal Institute of Technology Zurich, department of Mathematics.*

Morgan, J. P., (1994). RiskMetrics. (2nd edition), *J. P. Morgan.*

Neftci, S. (2000). Value at risk calculations, extreme events, and tail estimation. *The Journal of Derivatives,* 21, 1-15.

Owen, J. and Rabinovitch, R., (1983). On the Class of Elliptical Distributions and their Applications to the Theory of Portfolio Choice. *The Journal of Finance,* 38, 745-752

Pickands, J. I., (1975). Statistical inference using extreme value order statistics. *Annals of Statistics,* 3, 119-131.

Prescott, P., and Walden A. T., (1980). Maximum likelihood estimation of the parameters of generalized extreme value distribution, *Biometrika,* 67, 723-724.

Rachev, S.T., Mittnik, S. (2000). Stable Paretian Models in Finance. Wiley, New York, NY.

Rockafellar, R. T., and Uryasev, S., (2002). Conditional value at risk for general loss distribution. *Journal of Banking and Finance,* 26, 1443-1471.

Smith, R.L. (1985). Maximum likelihood estimation in a class of nonregular cases. *Biometrika*, 72, 67-90.

Szego, G., (2005). Measures of risk. *European Journal of Operational Research,* 163, 5-19.

Van Der Vaart, A.W. (1998). Asymptotic Statistics. Cambridge University Press, Cambridge.

Wilson, T., (1994). Plugging the GAP, *Risk,* 7, 74-80.

Winker, P., and Maringer, D., (2004). The hidden risks of optimizing bond portfolios under VaR. *Deutsche Bank Research, Frankfurt, Research Notes* 13.

Zhang, J., (2007). Likelihood moment estimation for Generalized Pareto Distribution. Aust. N. Z. J. Stat. 49(1), 69-77

Zikovic, S., (2008). Quantifying extreme risks in stock markets: A case of former Yugoslavian states. *Zb. rad. Ekon. fak. Rij.,* 26, 41-68

Gaurav Singh Chauhan is a faculty in the area of Finance and Accounting at Indian Institute of Management Indore. His research interests include Financial Regulation, Risk Management, Financial Economics and Corporate Finance. He is actively working in the field of Financial Stability and Risk Management as his research theme. He has recently presented his works in different International and National conferences where some of these are highly appreciated. He has recently published research articles related to Banking and Finance. He has worked in Oil and Gas industry for about five years in management positions, before joining IIM Indore.