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Vendors' View of Demand: A Platykurtic Class of Distributions

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Vendors' view of demand: a platykurtic class of distributions

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Abstract

A longstanding problem in demand analysis is to identify an appropriate demand distribution from qualitative feed-backs obtained from field survey. A typical survey from vendors would indicate towards a flat-top density curve tri-partitioned into positive region consisting of the most-likely set, boundary set that has possibility of belonging to the class, and negative region having least likelihood of occurrence. Such flat-top density curves, imply that multiple values are equally most likely to occur and hence all are modes. However, most popular probability models used in demand analysis are all uni-modal, leaving a single point in the positive region with maximum likelihood. In this paper, we propose a new class of probability distributions, called the stomped class of distributions, that provides better model fitting for the flat-top demand densities. We discuss the statistical properties of a special stomped distribution, called the stomped normal distribution. We have investigated the parameter estimation.

Keywords: Stomped normal distribution; Stochastic Demand; Demand distribution.

1. INTRODUCTION

One of the fundamental problems in demand survey from small vendors, like the green-grocers, is that the data comes in a qualitative format. Most common answer about future demand is an

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interval with a strong favor of the vendor. Further queries would reveal increasing dis-likelihood of the vendor as the possible demand moves away from the central interval of strongest likelihood. However, pin-pointing further within the central interval becomes very difficult. Since quantification of such a qualitative feedback on demand is difficult, the demand distribution is always assumed to be unimodal, though the qualitative feedback is hard to match with such assumptions. In this paper, we discuss a novel yet simple probability model that characterizes the type of qualitative feedback described above, using a modal set as opposed to the existing uni-modal demand distributions.

A careful study of typical qualitative responses on demand would reveal that the basis of such decision making pattern is a braod tri-partition of the randomness inherent to the unknown demand. In order to provide a structure to the unknown and stochastic demand, the range of possible demand is split into a *positive region* describing demand data-points that are most likely, a *boundary region* consisting of demand data-points that possibly can occur or somewhat likely, and a *negative region* consisting of the demand data-points that are lest likely to occur. However, the existing uni-modal probability models fail to incorporate this three-way decision structure for interpreting the demand distribution.

A resemblance with this three-way structure could be found in histograms with flat-top and high order contact. Such histograms are often encountered in probabilistic clustering of objects in image data analysis (Banerjee & Maji 2020; Banerjee & Maji 2019). Many other fields of studies also require probability models for explaining histograms depicting the tri-partition of data, including machine learning (Ghahramani 2015), computer vision (Marroquin, Mitter & Poggio 1987), data mining (Fang, Li, Jordan & Liu 2017), pattern recognition (Little 1993), web intelligence (Baldi, Frasconi & Smyth 2003), and so on. Even then, research on a suitable probability model in such cases has been limited to different mixtures of uni-modal distribution or their variations.

In this paper, we describe a novel class of distributions with a modal set in order to incorporate this specific three-way decision structure of the vendors, which gets disclosed during demand-surveys. Intuitively, a distribution with a modal set can be constructed from a uni-modal distribution by replacing the density of an interval containing the mode with that of a suitable uniform distribution, so that continuity of the probability density function (*pdf*) is maintained. Since the central region does not have a peak, and rather is flat, the new distribution could be thought of as a *headless or stomped* family of distributions. Clearly, stomped distributions with a modal set, provide robust alternatives for data distributions ranging from platykurtic to mesokurtic shapes.

Banerjee & Maji (2015) first introduced the idea of the stomped normal distribution and explained the effectiveness of such headless distributions for segmenting medical or brain MR images.

In particular, we present the theoretical results required to simulate from stomped normal distribution and moment computation. Due to lack of uni-modality, maximum likelihood (ML) estimation becomes infeasible. Hence, in this paper we use the method of moments estimation and show its effectiveness using simulation. The paper concludes with a discussion on the current work and its possible future direction.

2. STOMPED NORMAL DISTRIBUTION AND ITS PROPERTIES

In this section, we define stomped normal family and discuss its important statistical properties. We begin with the definition of the stomped normal distribution as follows:

Definition 2.1. Let X be a random variable over $\mathbb{R}$. X is said to follow one parameter stomped normal distribution (StN) iff its *pdf* is given as

$$f(x) = \frac{1}{D}\phi(z),\ x \in \mathbb{R};$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are *pdf* and CDF of standard normal distribution; $D = 2(\bar{\Phi}(k)+k\phi(k))$, where $\bar{\Phi}(\cdot)$ denotes the reliability or survival function of standard normal distribution, and

$$z = \begin{cases} k, & \text{if} \quad |x| < k \\ x, & \text{otherwise.} \end{cases}$$

We would use StN(k) as a notation for one parameter stomped normal distribution. The *pdf* plot of one parameter StN random variable is given below.

From the plot, it can be observed that the distribution has a modal set $(-0.5, 0.5)$, in which the density is maximum. If the underlying probability distribution of a feature in a cluster is StN, then the observed data could be assigned to the cluster if it belongs to the modal set. In addition, all these data points would have the same degree of belongingness, measured proportional to their densities, to the cluster. On the other hand, with a unimodal feature distribution, maximizing the data likelihood would point to a single value as a member to the cluster and all other nearby points will have decreased degree of belongingness to the cluster. The class of stomped normal distribution strictly includes the standard normal distribution at $k = 0$ (Azzalini 1985, see).

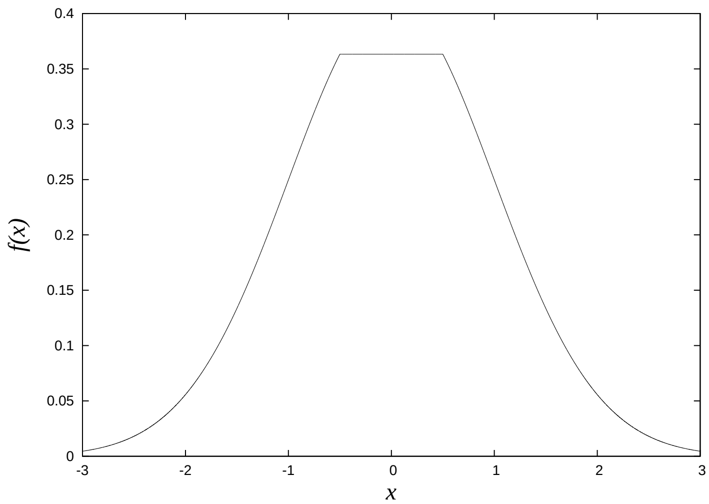

Figure 1: Probability density curve for StN distribution with $k = 0.5$

One parameter stomped normal distribution can be easily generalized to a class of three parameter probability distribution, which strictly includes normal distribution at a specific parameter setting. The general stomped normal distribution with three parameters is defined as follows.

Definition 2.2. Let X be a random variable over $\mathbb{R}$. X is said to follow three parameter stomped normal distribution (StN) iff its *pdf* is given as

$$f(x) = \frac{1}{D\sigma}\phi(z),\ x \in \mathbb{R}; \quad \text{where } D = 2(\bar{\Phi}(k) + k\phi(k)) \quad \text{and} \quad z = \begin{cases} k, & \text{if } \left|\frac{x-\mu}{\sigma}\right| < k \\ \frac{x-\mu}{\sigma}, & \text{otherwise.} \end{cases}$$

As in the case of single parameter stomped normal family, the three parameter distribution is also strictly inclusive of $N(\mu, \sigma^2)$ at $k = 0$. The following theorem provides the expression for the cumulative distribution function (CDF) of StN(μ, σ, k).

Theorem 2.1. *Let X be a real valued random variable following $StN(\mu, \sigma, k)$, $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, $k > 0$. The CDF of X is given by*

$$F(x) = \begin{cases} \frac{1}{D}\Phi\left(\frac{x-\mu}{\sigma}\right) & \textit{if } x \leq \mu - k\sigma \\ \frac{1}{2} + \frac{\phi(k)}{D}\left(\frac{x-\mu}{\sigma}\right) & \textit{if } \mu - k\sigma < x < \mu + k\sigma \\ 1 - \frac{1}{D}\Phi\left(\frac{\mu - x}{\sigma}\right) & \textit{if } x \geq \mu + k\sigma \end{cases} \tag{1}$$

Proof of the above theorem is available in Appendix A in the supplementary material. In case of single parameter stomped normal distribution, $StN(k)$, the CDF expression reduces to

$$F(x) = \begin{cases} \frac{1}{D}\Phi(x), & \text{if } \; x \leq -k \\ \frac{1}{2} + \frac{\phi(k)}{D}x, & \text{if } \; -k < x < k \\ 1 - \frac{1}{D}\Phi(-x), & \text{if } \; x \geq k \end{cases} \tag{2}$$

Also, from the above result, it can be observed that at $k = 0$, the CDF of three parameter stomped normal distribution reduces to a Gaussian one with mean μ and variance σ^2.

In what follows, we prove an important result for the stomped normal family. We show that this family has class preservation property with respect to linear transform, *i.e.* the linear transform of an StN random variable yields an StN random variable only.

Theorem 2.2. *If $X \sim StN(\mu, \sigma, k)$, then $a + bX \sim StN(a + b\mu, |b|\sigma, k)$, where $b \neq 0$.*

For the proof of the above theorem, see Appendix A in the supplementary material.

2.1 Simulating from stomped normal distribution

Using the result in Theorem 2.2, we can simulate from stomped normal distribution using the following inverse transformation. Let u be a random number drawn from $Unif(0, 1)$. Then, a single parameter stomped normal variate x is generated by the inverse of F on u as below:

$$x = F^{-1}(u) = \begin{cases} \Phi^{-1}(Du) & \text{if } \; u \leq \frac{1}{D}\Phi(-k) \\ \frac{D}{\phi(k)}\left(u - \frac{1}{2}\right) & \text{if } \; \frac{1}{D}\Phi(-k) < u < 1 - \frac{1}{D}\Phi(-k) \\ \Phi^{-1}\left(D(1-u)\right) & \text{if } \; u \geq 1 - \frac{1}{D}\Phi(-k) \end{cases} \tag{3}$$

To draw from StN(μ, σ, k), we note from Theorem 2.2 that $Y = \mu + \sigma X \sim StN(\mu, \sigma, k)$. Therefore, by linear transformation of X, a random sample from the three parameter stomped normal distribution can be obtained.

2.2 Moment generating function and moments

Existence of the moment generating function (MGF) of stomped normal distribution follows from the fact that it is dominated by a normal distribution. Thus all moments of stomped normal distribution would exist. In the following theorem we provide the expression of its MGF.

Theorem 2.3. *Let X be a real valued random variable following stomped normal distribution $StN(\mu, \sigma, k)$. Then the moment generating function of X is given by:*

$$M_X(t) = \frac{e^{t\mu}}{D}\left[e^{\frac{t^2\sigma^2}{2}}\Big(\Phi(-k - t\sigma) + \Phi(-k + t\sigma)\Big) + 2\phi(k)\sum_{i=0}^{\infty}\frac{(t\sigma)^{2i}k^{2i+1}}{(2i+1)!}\right]. \tag{4}$$

For proof of the above theorem, see Appendix A in the supplementary material. Differentiating the MGF, we can obtain different raw moments of the stomped normal distribution. Thus,

$$\begin{aligned}
E[X] &= \frac{\partial}{\partial t}M_X(t)\Big|_{t=0} = \mu + \frac{1}{\sqrt{2\pi}D}e^{t\mu}\Bigg[e^{\frac{t^2\sigma^2}{2}}t\sigma^2\sqrt{2\pi}\left(\Phi(-k - t\sigma) + \Phi(-k + t\sigma)\right) \\
&\quad + e^{\frac{t^2\sigma^2}{2}}\sqrt{2\pi}\left(-\sigma\phi(-k - t\sigma) + \sigma\phi(-k + t\sigma)\right) + 2e^{-\frac{k^2}{2}}\sum_{i=1}^{\infty}\frac{(t\sigma)^{2i-1}k^{2i+1}2i\sigma}{(2i+1)!}\Bigg]\Bigg|_{t=0} \\
&= \mu
\end{aligned} \tag{5}$$

However, it seems difficult to get higher order moments, specially the central ones, by differentiating the MGF. In what follows, we provide a recursion relation for the central moments of the StN distribution. We begin with the following two important lemmas in this context.

Lemma 2.4. *Consider a real valued random variable $X \sim StN(\mu, \sigma, k)$ with MGF $M_X(t)$. Denote the partial derivative with respect to t by the operator $\bigtriangledown$ and the identity operator by $\tilde{I}$. Then,*

$$(\bigtriangledown - \mu\tilde{I})^p\ (tM_X(t)) = t(\bigtriangledown - \mu\tilde{I})^p\ M_X(t) + p(\bigtriangledown - \mu\tilde{I})^{p-1}\ M_X(t). \tag{6}$$

Proof. Let us first consider the result for $p = 1$.

$$\left(\bigtriangledown - \mu\tilde{I}\right)\left(tM_X(t)\right) = t\frac{\partial}{\partial t}M_X(t) + M_X(t) - \mu t M_X(t) = t\left(\bigtriangledown - \mu\tilde{I}\right)M_X(t) + M_X(t).$$

Next, for $p = 2$,

$$\begin{aligned}
\left(\bigtriangledown - \mu\tilde{I}\right)^2\left(tM_X(t)\right) &= \left(\bigtriangledown - \mu\tilde{I}\right)\left(t\left(\bigtriangledown - \mu\tilde{I}\right)M_X(t) + M_X(t)\right), \quad \text{[using above result for } p = 1] \\
&= t\left(\bigtriangledown - \mu\tilde{I}\right)^2 M_X(t) + \left(\bigtriangledown - \mu\tilde{I}\right)M_X(t) + \left(\bigtriangledown - \mu\tilde{I}\right)M_X(t) \\
&= t\left(\bigtriangledown - \mu\tilde{I}\right)^2 M_X(t) + 2\left(\bigtriangledown - \mu\tilde{I}\right)M_X(t).
\end{aligned}$$

Hence, the proof follows from induction. □

It can be noted here that $\left(\bigtriangledown - \mu\tilde{I}\right)^p M_X(t)\ |_{t=0} = \mu_p$, where μ_p is the p^{th} order central moment of $X \sim StN(\mu, \sigma, k)$. Thus, from the above lemma we get, $(\bigtriangledown - \mu\tilde{I}\ )^p\ (tM_X(t))\ |_{t=0} = p\mu_{p-1}$.

Next, we define, $F_0(t) = \frac{2\sigma}{D}\phi(k)\sum_{i=1}^{\infty}\frac{2i(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}$, $F_1(t) = e^{t\mu}F_0(t)$, and $F_p(t) = (\triangledown - \mu\tilde{I})F_{p-1}(t)$, $\forall p \geq 2$, and $t \in \mathbb{R}$. Differentiating the MGF in Theorem 2.3, it can be easily shown that

$$M_X'(t) = (\mu + t\sigma^2)M_X(t) + \frac{2\sigma}{D}e^{t\mu}\phi(k)\sum_{i=1}^{\infty}\frac{2i(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!} = (\mu + t\sigma^2)M_X(t) + F_1(t). \tag{7}$$

In the following lemma, we establish the relation between the MGF of $StN(\mu, \sigma, k)$ and $F_p(t)$.

Lemma 2.5. *Consider a random variable $X \sim StN(\mu, \sigma, k)$. Then, for any $t \in \mathbb{R}$*

$$\begin{aligned} F_{p+1}(t) &= 2e^{t\mu}\frac{\sigma}{D}\phi(k)J_p(t) \\ &= \left(\triangledown - \mu\tilde{I}\right)^{p+1}M_X(t) - \sigma^2\left(t\left(\triangledown - \mu\tilde{I}\right)^{p}M_X(t) + p\left(\triangledown - \mu\tilde{I}\right)^{p-1}M_X(t)\right) \end{aligned} \tag{8}$$

where $J_p(t) = \frac{\partial^p}{\partial t^p}\left(\sum_{i=1}^{\infty}\frac{2i(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}\right)$.

For the proof of this lemma, see Appendix A in the supplementary material. In the light of the above two lemmas, we now provide a recursion relation of the central moments of $StN(\mu, \sigma, k)$ in the following theorem.

Theorem 2.6. *The central moments of an $StN(\mu, \sigma, k)$ random variable follows the recursion relation:*

$$\mu_{p+1} = \begin{cases} p\sigma^2\mu_{p-1} + 2\frac{\phi(k)}{D}\frac{\sigma^{p+1}k^{p+2}}{(p+2)}, & \text{if } \ p = 2d-1 \\ 0, & \text{if } \ p = 2d \end{cases} \tag{9}$$

where d is any positive integer.

Detailed proof of the above theorem is provided in Appendix A in the supplementary material. It follows from the above theorem that the variance of $StN(\mu, \sigma, k)$ is

$$\mu_2 = \sigma^2 + \frac{2\phi(k)}{D}\sigma^2\frac{k^3}{3} = \sigma^2\left(1 + \frac{2\phi(k)}{D}\frac{k^3}{3}\right) \text{ [assuming } d = 1]. \tag{10}$$

Further, the fourth central moment is given by

$$\mu_4 = 3\sigma^4 + \frac{2\phi(k)}{D}\sigma^4\left(k^3 + \frac{k^5}{5}\right) = \sigma^4\left(3 + \frac{2\phi(k)}{D}\left(k^3 + \frac{k^5}{5}\right)\right). \tag{11}$$

As $k \to 0$, $\mu_4 \to 3\sigma^4$, indicating $StN(\mu, \sigma, k)$ resembles $N(\mu, \sigma^2)$ more and more closely as the truncation parameter approaches zero. In general, solving the recursion in (9), the even order moment can be expressed as

$$
\begin{aligned}
\mu_{2d} &= (2d-1)\sigma^2\mu_{2d-2} + 2\frac{\phi(k)}{D}\sigma^{2d}\frac{k^{2d+1}}{2d+1} \\
&= (2d-1)(2d-3)\sigma^4\mu_{2d-4} + 2\frac{\phi(k)}{D}\sigma^{2d}\left(k^{2d-1} + \frac{k^{2d+1}}{2d+1}\right) \\
&= (2d-1)(2d-3)(2d-5)\sigma^6\mu_{2d-6} + 2\frac{\phi(k)}{D}\sigma^{2d}\left((2d-1)k^{2d-3} + k^{2d-1} + \frac{k^{2d+1}}{2d+1}\right) \\
& \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&= \frac{(2d-1)!}{2^{d-1}(d-1)!}\,\sigma^{2d} + \frac{2\phi(k)}{D}\sigma^{2d}\Bigg(\frac{(2d-1)!}{2^{d-1}(d-1)!3}k^3 + \frac{2\ (2d-1)!\ 2!}{2^{d-1}(d-1)!5!}k^5 + (2d-1)k^{2d-3} \\
&\quad + k^{2d-1} + \frac{k^{2d+1}}{2d+1}\Bigg). \qquad (12)
\end{aligned}
$$

3. PARAMETER ESTIMATION FOR STOMPED NORMAL DISTRIBUTION

In this section we discuss the parameter estimation method of stomped normal distribution. The stomped normal density is uniform over the range $(\mu - k\sigma, \mu + k\sigma)$ and hence, there is no unique maxima in that interval, which in turn indicates maximum likelihood estimation of μ is not possible. We rather consider two other methods of estimation. First, we use the method of moments to estimate the parameters μ, σ, and k. The second approach exploits the symmetry to estimate μ iteratively, and the remaining parameters are estimated using an iterative optimization scheme. In both cases, we use simulation to show the accuracy of the estimators.

3.1 Method of moment estimation

In the method of moment (MoM) estimation, the estimating equations are formed by equating the sample and population moments. Equating the first three non-zero moments, *viz.* 1^{st}, 2^{nd} and 4^{th}, to their sample counter parts, we obtain the following three equations:

$$\bar{x} = \mu, \qquad (13)$$

$$s^2 = \sigma^2\left(1 + \frac{2\phi(k)}{D}\frac{k^3}{3}\right), \qquad (14)$$

and $$m_4 = \sigma^4\left(3 + \frac{2\phi(k)}{D}\left(k^3 + \frac{k^5}{5}\right)\right), \qquad (15)$$

where, $\bar{x}$, s^2, and m_4 are the sample mean, variance, and 4^{th} central moment, respectively. The parameter μ is estimated from (13), while k and σ are estimated from (14) and (15). After substituting σ^2 from (14) into (15), k is estimated numerically from the implicit equation. Using the estimated value of k in (14), we find the MoM estimator of σ.

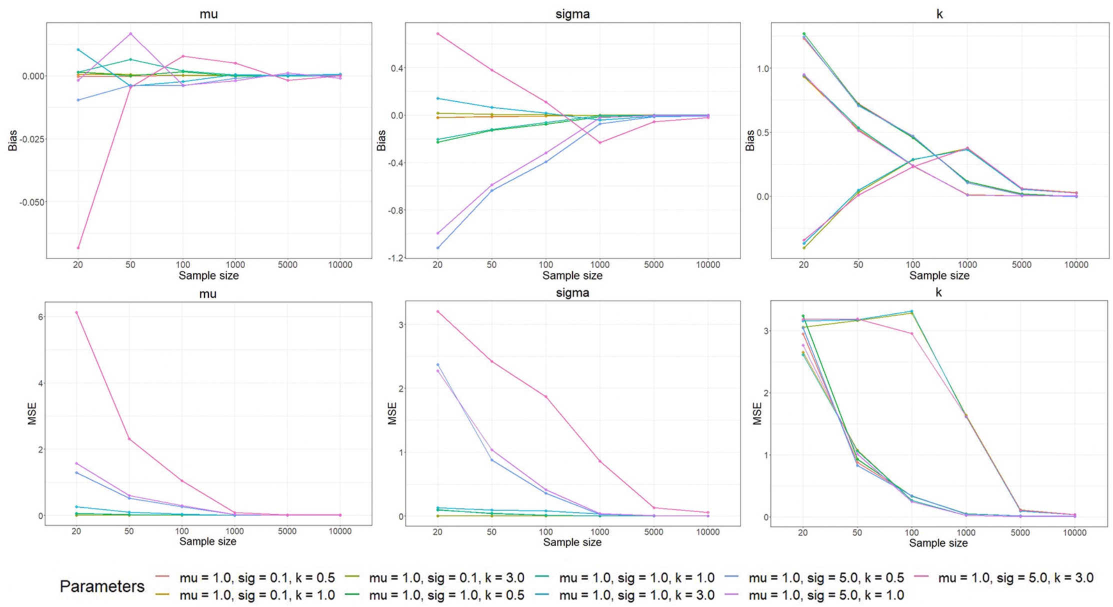

Figure 2: Estimation bias and mean-square error (MSE) plots of the method of moment estimators of the parameters μ, σ, and k.

The asymptotic unbiasedness of the estimators could be intuitively argued, noting that sample moments converge in L_2 to the population moments. Here, we inspect the bias and mean square errors (MSE) of the parameter estimates for increasing sample sizes to gauge the performance of the MoM estimators. A random sample of size n is simulated from an StN distribution with known μ, σ, and k, where $n \in \{20, 50, 100, 1000, 5000, 10000\}$. The MoM parameter estimates are obtained by solving (13)-(15). For each sample size, this process is repeated $M = 5000$ times to compute bias and MSE of the estimates. In this work, we have considered a fixed location parameter $\mu = 1$ and $\sigma \in \{0.1, 1, 5\}$, $k \in \{0.5, 1, 3\}$ for illustration purpose. Bias and MSE of the MoM estimators of μ, σ, and k are reported in Fig. 2 corresponding to different sets of values.

It can be observed from the graphs that the MoM estimators are asymptotically unbiased. But the convergence happens at different rates for different parameters. For example, $\hat{\mu}$ is asymptotically unbiased for almost all combinations with the most poor convergence occurring for $(\mu, \sigma, k) = (1, 5, 3)$ for both bias and MSE plots, *i.e.* when σ and k are both high in this set up. In a similar manner, the MoM estimators of σ and k converge fast except when the corresponding true values are high, *i.e.* $\sigma = 5$ and $k = 3$.

4. DISCUSSION

In this paper, we have provided an elicitation of probability model from typical qualitative feedback from demand survey. In particular we have motivated the problem of finding a suitable demand distribution for responses indicating flat-top histograms using stomped family of distributions with special reference to stomped normal distribution. Such stomped distributions facilitate characterization of the tri-partition of data into three classes, *viz.* the positive region consisting of the data-points that equally most-likely, boundary region with fast decreasing likelihood of demand, and negative region consisting of the demand-set that is not likely to occur. Our main contribution in this paper is to provide a class of platykurtic distributions containing normal, which characterizes the tri-partitioned decision structure derived from qualitative demand survey feedback. Here, we have provided major statistical properties of this novel distribution with varying peakedness, ranging from platykurtic to mesokurtic distributions, with a focus on stomped normal distribution. We have also investigated the parameter estimation method for stomped normal family, which lacks the advantage of ML estimation. For this purpose, we have performed an extensive simulation study over 9 different parameter combinations and 6 different sample sizes.

REFERENCES

Amaratunga, D., & Cabrera, J. (2001), "Analysis of Data From Viral DNA Microchips," *Journal of the American Statistical Association*, 96(456), 1161–1170.

Aykroyd, R. G., & Zimeras, S. (1999), "Inhomogeneous Prior Models for Image Reconstruction," *Journal of the American Statistical Association*, 94(447), 934–946.

Azzalini, A. (1985), "A Class of Distributions Which Includes the Normal Ones," *Scandinavian Journal of Statistics*, 12(2), 171–178.

Baldi, P., Frasconi, P., & Smyth, P. (2003), *Modeling the Internet and the Web: Probabilistic Methods and Algorithms* Wiley.

Banerjee, A., & Maji, P. (2015), "Rough Sets and Stomped Normal Distribution for Simultaneous Segmentation and Bias Field Correction in Brain MR Images," *IEEE Transactions on Image Processing*, 24(12), 5764–5776.

Banerjee, A., & Maji, P. (2016), "Rough-Probabilistic Clustering and Hidden Markov Random Field Model for Segmentation of HEp-2 Cell and Brain MR Images," *Applied Soft Computing*, 46, 558–576.

Banerjee, A., & Maji, P. (2018), "Spatially Constrained Student's t-Distribution Based Mixture Model for Robust Image Segmentation," *Journal of Mathematical Imaging and Vision*, 60(3), 355–381.

Banerjee, A., & Maji, P. (2019), "Segmentation of Bias Field Induced Brain MR Images Using Rough Sets and Stomped-t distribution," *Information Sciences*, 504, 520–545.

Banerjee, A., & Maji, P. (2020), "A Spatially Constrained Probabilistic Model for Robust Image Segmentation," *IEEE Transactions on Image Processing*, 29, 4898–4910.

Beheshti, I., Sone, D., Farokhian, F., Maikusa, N., & Matsuda, H. (2018), "Gray Matter and White Matter Abnormalities in Temporal Lobe Epilepsy Patients with and without Hippocampal Sclerosis," *Frontiers in Neurology*, 9, 107.

Besag, J. (1986), "On the Statistical Analysis of Dirty Pictures," *Journal of the Royal Statistical Society, Series B*, 48(3), 259–302.

Fang, E. X., Li, M.-D., Jordan, M. I., & Liu, H. (2017), "Mining Massive Amounts of Genomic Data: A Semiparametric Topic Modeling Approach," *Journal of the American Statistical Association*, 112(519), 921–932.

Feng, X., Li, T., Song, X., & Zhu, H. (2020), "Bayesian Scalar on Image Regression With Nonignorable Nonresponse," *Journal of the American Statistical Association*, 115(532), 1574–1597.

Forbes, F., & Raftery, A. E. (1999), "Bayesian Morphology: Fast Unsupervised Bayesian Image Analysis," *Journal of the American Statistical Association*, 94(446), 555–568.

Fraley, C., & Raftery, A. E. (2002), "Model-Based Clustering, Discriminant Analysis, and Density Estimation," *Journal of the American Statistical Association*, 97(458), 611–631.

García-Lorenzo, D., Francis, S., Narayanan, S., Arnold, D. L., & Collins, D. L. (2013), "Review of Automatic Segmentation Methods of Multiple Sclerosis White Matter Lesions on Conventional Magnetic Resonance Imaging," *Medical Image Analysis*, 17(1), 1–18.

Ghahramani, Z. (2015), "Probabilistic Machine Learning and Artificial Intelligence," *Nature*, 521(7553), 452–459.

Greenspan, H., Ruf, A., & Goldberger, J. (2006), "Constrained Gaussian Mixture Model Framework for Automatic Segmentation of MR Brain Images," *IEEE Transactions on Medical Imaging*, 25(9), 1233–1245.

Jansen, M., & Bultheel, A. (2001), "Empirical Bayes Approach to Improve Wavelet Thresholding for Image Noise Reduction," *Journal of the American Statistical Association*, 96(454), 629–639.

Johnson, V. E. (1994), "A Model for Segmentation and Analysis of Noisy Images," *Journal of the American Statistical Association*, 89(425), 230–241.

Kenobi, K., Dryden, I. L., & Le, H. (2010), "Shape Curves and Geodesic Modelling," *Biometrika*, 97(3), 567–584.

Kim, G.-W., Kim, Y.-H., & Jeong, G.-W. (2017), "Whole Brain Volume Changes and Its Correlation with Clinical Symptom Severity in Patients with Schizophrenia: A DARTEL-Based VBM Study," *PLOS ONE*, 12(5), 1–15.

Kolaczyk, E. D., Ju, J., & Gopal, S. (2005), "Multiscale, Multigranular Statistical Image Segmentation," *Journal of the American Statistical Association*, 100(472), 1358–1369.

Liang, Z., MacFall, J. R., & Harrington, D. P. (1994), "Parameter Estimation and Tissue Segmentation from Multispectral MR Images," *IEEE Transactions on Medical Imaging*, 13(3), 441–449.

Little, R. J. A. (1993), "Pattern-Mixture Models for Multivariate Incomplete Data," *Journal of the American Statistical Association*, 88(421), 125–134.

Marroquin, J., Mitter, S., & Poggio, T. (1987), "Probabilistic Solution of Ill-Posed Problems in Computational Vision," *Journal of the American Statistical Association*, 82(397), 76–89.

Miglioretti, D. L., McCulloch, C., & Zeger, S. L. (2002), "Combining Images Across Multiple Subjects," *Journal of the American Statistical Association*, 97(457), 125–135.

Müller, P., & Rosner, G. L. (1997), "A Bayesian Population Model with Hierarchical Mixture Priors Applied to Blood Count Data," *Journal of the American Statistical Association*, 92(440), 1279–1292.

Nguyen, H., Cressie, N., & Braverman, A. (2012), "Spatial Statistical Data Fusion for Remote Sensing Applications," *Journal of the American Statistical Association*, 107(499), 1004–1018.

Nguyen, T. M., Wu, Q. M. J., & Ahuja, S. (2010), "An Extension of the Standard Mixture Model for Image Segmentation," *IEEE Transactions on Neural Networks*, 21(8), 1326–1338.

Petersen, J., Jäger, P. F., Isensee, F., Kohl, S. A. A. et al. (2019), Deep Probabilistic Modeling of Glioma Growth,, in *Proceedings of the Medical Image Computing and Computer Assisted Intervention – MICCAI 2019*, Springer International Publishing, pp. 806–814.

Qiu, P. (2007), "Jump Surface Estimation, Edge Detection, and Image Restoration," *Journal of the American Statistical Association*, 102(478), 745–756.

Sfikas, G., Nikou, C., & Galatsanos, N. (2007), Robust Image Segmentation with Mixtures of Students's t-Distribution,, in *Proceedings of the IEEE International Conference on Image Processing*, pp. 273–276.

Tang, X., Bi, X., & Qu, A. (2020), "Individualized Multilayer Tensor Learning With an Application in Imaging Analysis," *Journal of the American Statistical Association*, 115(530), 836–851.

Wang, Y., & Blei, D. M. (2019), "The Blessings of Multiple Causes," *Journal of the American Statistical Association*, 114(528), 1574–1596.

Wells, III, W. M., Grimson, W. E. L., Kikins, R., & Jolezs, F. A. (1996), "Adaptive Segmentation of MRI Data," *IEEE Transactions on Medical Imaging*, 15(8), 429–442.

Zhu, H., Li, Y., Ibrahim, J. G., Shi, X. et al. (2009), "Regression Models for Identifying Noise Sources in Magnetic Resonance Images," *Journal of the American Statistical Association*, 104(486), 623–637.

Zhu, H., Zhang, H., Ibrahim, J. G., & Peterson, B. S. (2007), "Statistical Analysis of Diffusion Tensors in Diffusion-Weighted Magnetic Resonance Imaging Data," *Journal of the American Statistical Association*, 102(480), 1085–1102.

Appendix A PROOFS OF THEOREMS

A.1 Proof of Theorem 2.1

Proof. We first consider the case $x \leq \mu - k\sigma$. The CDF in this region is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du = \int_{-\infty}^{x} \frac{1}{D}\frac{1}{\sigma}\phi\left(\frac{u-\mu}{\sigma}\right) du = \frac{1}{D}\Phi\left(\frac{x-\mu}{\sigma}\right). \quad (16)$$

Next consider $\mu - k\sigma < x < \mu + k\sigma$. In this case the CDF is given by

$$\begin{aligned} F(x) = P(X \leq x) &= P(X \leq \mu - k\sigma) + P(\mu - k\sigma < X \leq x) \\ &= \frac{1}{D}(1 - \Phi(k)) + \frac{1}{D}\frac{1}{\sigma}\phi(k)(x - (\mu - k\sigma)) \\ &= \frac{1}{D}\left(1 - \Phi(k) + \left(\frac{x-\mu}{\sigma}\right)\phi(k) + k\phi(k)\right) \\ &= \frac{1}{D}\left(\frac{D}{2} + \left(\frac{x-\mu}{\sigma}\right)\phi(k)\right) = \frac{1}{2} + \frac{\phi(k)}{D}\left(\frac{x-\mu}{\sigma}\right). \end{aligned} \quad (17)$$

For $x \geq \mu + k\sigma$, the CDF is given by

$$\begin{aligned} F(x) = P(X \leq x) &= P(X \leq \mu - k\sigma) + P(\mu - k\sigma < X < \mu + k\sigma) + P(\mu + k\sigma \leq X \leq x) \\ &= \frac{1}{D}(1 - \Phi(k)) + \frac{1}{D}2k\phi(k) + \int_{\mu+k\sigma}^{x} f(u)du \\ &= \frac{1}{D}(1 - \Phi(k)) + \frac{1}{D}2k\phi(k) + \int_{\mu+k\sigma}^{x} \frac{1}{D\sigma}\phi\left(\frac{u-\mu}{\sigma}\right) du \\ &= \frac{1}{D}\left[1 - \Phi(k) + 2k\phi(k) + \Phi\left(\frac{x-\mu}{\sigma}\right) - \Phi(k)\right] \\ &= \frac{1}{D}\left[1 - 2\Phi(k) + 2k\phi(k) + \Phi\left(\frac{x-\mu}{\sigma}\right)\right] \\ &= \frac{1}{D}\left[D - \Phi\left(-\frac{x-\mu}{\sigma}\right)\right] = 1 - \frac{1}{D}\Phi\left(\frac{\mu-x}{\sigma}\right). \end{aligned} \quad (18)$$

$$\text{Hence,} \quad \text{CDF :} \quad F(x) = \begin{cases} \frac{1}{D}\Phi\left(\frac{x-\mu}{\sigma}\right) & \text{if } x \leq \mu - k\sigma \\ \frac{1}{2} + \frac{\phi(k)}{D}\left(\frac{x-\mu}{\sigma}\right) & \text{if } \mu - k\sigma < x < \mu + k\sigma \\ 1 - \frac{1}{D}\Phi\left(\frac{\mu-x}{\sigma}\right) & \text{if } x \geq \mu + k\sigma \end{cases} \quad (19)$$

□

A.2 Proof of Theorem 2.2

Proof. Let us consider a linear transformation of $X \sim StN(\mu, \sigma, k)$ as $Y = a + bX$. To derive the probability distribution of Y, we consider two cases, *viz.* $b > 0$ and $b < 0$, separately.

Case 1: $b > 0$

$$F_Y(y) = P(a + bX \leq y) = F_X\left(\frac{y-a}{b}\right)$$

$$= \begin{cases} \frac{1}{D}\Phi\left(\frac{\frac{y-a}{b} - \mu}{\sigma}\right) & \text{if } \frac{y-a}{b} \leq \mu - k\sigma \\ \frac{1}{2} + \frac{\phi(k)}{D}\left(\frac{\frac{y-a}{b} - \mu}{\sigma}\right) & \text{if } \mu - k\sigma < \frac{y-a}{b} < \mu + k\sigma \\ 1 - \frac{1}{D}\Phi\left(-\frac{\frac{y-a}{b} - \mu}{\sigma}\right) & \text{if } \frac{y-a}{b} \geq \mu + k\sigma \end{cases} \tag{20}$$

$$\Rightarrow F_Y(y) = \begin{cases} \frac{1}{D}\Phi\left(\frac{y - (a + b\mu)}{b\sigma}\right) & \text{if } y \leq a + b\mu - kb\sigma \\ \frac{1}{2} + \frac{\phi(k)}{D}\left(\frac{y - (a + b\mu)}{b\sigma}\right) & \text{if } a + b\mu - kb\sigma < y < a + b\mu + kb\sigma \\ 1 - \frac{1}{D}\Phi\left(-\frac{y - (a + b\mu)}{b\sigma}\right) & \text{if } y \geq a + b\mu + kb\sigma \end{cases} \tag{21}$$

Hence, $Y = a + bX \sim StN(a + b\mu, b\sigma, k)$, for $b > 0$.

Case 2: $b < 0$

$$F_Y(y) = P(a + bX \leq y) = 1 - F_X\left(\frac{y-a}{b}\right) = 1 - F_X\left(\frac{a-y}{\mid b \mid}\right)$$

$$= \begin{cases} 1 - \frac{1}{D}\Phi\left(\frac{\frac{a-y}{b} - \mu}{\sigma}\right) & \text{if } \frac{a-y}{b} \leq \mu - k\sigma \\ \frac{1}{2} - \frac{\phi(k)}{D}\left(\frac{\frac{a-y}{b} - \mu}{\sigma}\right) & \text{if } \mu - k\sigma < \frac{a-y}{b} < \mu + k\sigma \\ \frac{1}{D}\Phi\left(\frac{\frac{a-y}{b} - \mu}{\sigma}\right) & \text{if } \frac{a-y}{b} \geq \mu + k\sigma \end{cases} \tag{22}$$

$$\Rightarrow F_Y(y) = \begin{cases} \frac{1}{D}\Phi\left(\frac{y - (a + b\mu)}{|b|\sigma}\right), & \text{if } y \leq (a + b\mu) - k|b|\sigma \\ \frac{1}{2} + \frac{\phi(k)}{D}\left(\frac{y - (a + b\mu)}{|b|\sigma}\right) & \text{if } (a + b\mu) - k|b|\sigma < y < (a + b\mu) + k|b|\sigma \\ 1 - \frac{1}{D}\Phi\left(-\frac{y - (a + b\mu)}{|b|\sigma}\right) & \text{if } y \geq (a + b\mu) + k|b|\sigma \end{cases} \tag{23}$$

Hence, $Y = a + bX \sim StN(a + b\mu, |b|\sigma, k)$, where $b < 0$. Combining the results from Case 1 and 2 we get, $Y = a + bX \sim StN(a + b\mu, |b|\sigma, k)$, where $b \neq 0$. □

A.3 Proof of Theorem 2.3

Proof.

$$
\begin{aligned}
M_X(t) = E\left[e^{tX}\right] &= \int_{-\infty}^{-k} e^{tx} f(x)dx + \int_{-k}^{k} e^{tx} f(x)dx + \int_{k}^{\infty} e^{tx} f(x)dx \\
&= \frac{1}{D}\left[\int_{-\infty}^{-k} e^{tx}\phi\left(\frac{x-\mu}{\sigma}\right)dx + \int_{-k}^{k} e^{tx}\phi(k)dx + \int_{k}^{\infty} e^{tx}\phi(x)dx\right] \\
&= \frac{e^{t\mu}}{D\sqrt{2\pi}}\left[\int_{-\infty}^{-k} e^{t\sigma z - \frac{z^2}{2}}dz + \int_{-k}^{k} e^{t\sigma z - \frac{k^2}{2}}dz + \int_{k}^{\infty} e^{t\sigma z - \frac{z^2}{2}}dz\right] \\
&= \frac{1}{D}e^{t\mu + \frac{t^2\sigma^2}{2}}\left[\Phi(-k + t\sigma) + \Phi(-k - t\sigma) + \frac{1}{t\sigma}(\phi(k - t\sigma) - \phi(k + t\sigma))\right]. \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
\text{Now,}\quad \frac{1}{t\sigma}(e^{tk\sigma} - e^{-tk\sigma}) &= \frac{1}{p}(e^{pk} - e^{-pk}) = \frac{1}{p}2\left(pk + \frac{(pk)^3}{3!} + \frac{(pk)^5}{5!} + \cdots\right) \\
&= 2\left(k + \frac{p^2k^3}{3!} + \frac{p^4k^5}{5!} + \cdots\right) = 2\sum_{i=0}^{\infty}\frac{(t\sigma)^{2i}k^{2i+1}}{(2i+1)!}. \qquad (25)
\end{aligned}
$$

Applying (25) into (24), we get

$$
M_X(t) = \frac{e^{t\mu}}{D}\left[e^{\frac{t^2\sigma^2}{2}}\Big(\Phi(-k - t\sigma) + \Phi(-k + t\sigma)\Big) + \sqrt{\frac{2}{\pi}}e^{-\frac{k^2}{2}}\sum_{i=0}^{\infty}\frac{(t\sigma)^{2i}k^{2i+1}}{(2i+1)!}\right].
$$

□

A.4 Proof of Lemma 2.5

Proof. For $X \sim StN(\mu, \sigma, k)$,

$$F_1(t) = 2e^{t\mu}\frac{1}{D}\phi(k)\sum_{i=1}^{\infty}\frac{2i\sigma(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}, \text{ [from Eq. (10) in main paper]}$$

$$\Rightarrow \frac{\partial}{\partial t}F_1(t) = 2e^{t\mu}\mu\frac{1}{D}\phi(k)\sum_{i=1}^{\infty}\frac{2i\sigma(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!} + 2e^{t\mu}\frac{1}{D}\phi(k)\frac{\partial}{\partial t}\left(\sum_{i=1}^{\infty}\frac{2i\sigma(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}\right)$$

$$\Rightarrow \frac{\partial}{\partial t}F_1(t) - \mu F_1(t) = F_2(t) = 2e^{t\mu}\frac{1}{D}\phi(k)\frac{\partial}{\partial t}\left(\sum_{i=1}^{\infty}\frac{2i\sigma(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}\right).$$

Repeating the same steps, we get

$$F_{p+1}(t) = C\sigma J_p(t), \text{ where } J_p(t) = \frac{\partial^p}{\partial t^p}\left(\sum_{i=1}^{\infty}\frac{2i(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}\right) \text{ and } C = 2e^{t\mu}\frac{\phi(k)}{D}. \tag{26}$$

Also,

$$\begin{aligned} F_{p+1}(t) &= \left(\triangledown - \mu\tilde{I}\right)F_p(t) = \left(\triangledown - \mu\tilde{I}\right)^2 F_{p-1}(t) = \cdots = \left(\triangledown - \mu\tilde{I}\right)^p F_1(t) \\ &= \left(\triangledown - \mu\tilde{I}\right)^p\left(\triangledown - (\mu + t\sigma^2)\tilde{I}\right)M_X(t) \quad \text{[from Eq. (10) in main paper]} \\ &= \left(\triangledown - \mu\tilde{I}\right)^{p+1}M_X(t) - \sigma^2\left(\triangledown - \mu\tilde{I}\right)^p\left(tM_X(t)\right). \end{aligned} \tag{27}$$

Finally, from Lemma 2.4 and Eqns. (26) and (27), we get

$$\begin{aligned} F_{p+1}(t) &= 2e^{t\mu}\frac{\phi(k)}{D}\sigma J_p(t) \\ &= \left(\triangledown - \mu\tilde{I}\right)^{p+1}M_X(t) - \sigma^2\left(t\left(\triangledown - \mu\tilde{I}\right)^p M_X(t) + p\left(\triangledown - \mu\tilde{I}\right)^{p-1}M_X(t)\right). \end{aligned}$$

□

A.5 Proof of Theorem 2.6

Proof. From Lemma 2.5, it follows that

$$F_{p+1}(0) = 2\frac{\phi(k)}{D}\sigma J_p(0) = \mu_{p+1} - p\sigma^2\mu_{p-1}. \tag{28}$$

Further, from Eq. (10) in main paper, it can be noticed that

$$\begin{aligned}
M'_X(t) &= F_1(t) + (\mu + t\sigma^2)M_X(t) \\
\therefore \frac{\partial}{\partial t}F_1(t) &= M''_X(t) - \sigma^2 M_X(t) - (\mu + t\sigma^2)M'_X(t) \\
&= M''_X(t) - \sigma^2\left[F_1(t) + (\mu + t\sigma^2)M_X(t)\right] \\
\Rightarrow M''_X(t) &= \frac{\partial}{\partial t}F_1(t) + \sigma^2 M'_X(t).
\end{aligned} \tag{29}$$

Differentiating the above equation $(p-2)$ times we get,

$$M_X^{(p)}(t) = \frac{\partial^{p-1}}{\partial t^{p-1}}F_1(t) + \sigma^2 M_X^{(p-1)}(t) \tag{30}$$

where, $f^{(p)}(x) = \frac{\partial^p}{\partial x^p}f(x)$.

Further notice,

$$\begin{aligned}
J_p(t) = \frac{\partial^p}{\partial t^p}\left(\sum_{i=1}^{\infty}\frac{2i(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}\right) &= \frac{\partial^{p-1}}{\partial t^{p-1}}\frac{\partial}{\partial t}\left(\sum_{i=1}^{\infty}\frac{2i(t\sigma)^{2i-1}k^{2i+1}}{(2i+1)!}\right) \\
&= \frac{\partial^{p-1}}{\partial t^{p-1}}\left(\sum_{i=1}^{\infty}\frac{2i(2i-1)\sigma(t\sigma)^{2i-2}k^{2i+1}}{(2i+1)!}\right) \\
&= \frac{\partial^{p-1}}{\partial t^{p-1}}\left(\sum_{i=0}^{\infty}\frac{1}{(2i+3)(2i)!}\sigma(t\sigma)^{2i}k^{2i+3}\right) \\
&= \frac{\partial^{p-2}}{\partial t^{p-2}}\left(\sum_{i=1}^{\infty}\frac{1}{(2i+3)(2i-1)!}\sigma^2(t\sigma)^{2i-1}k^{2i+3}\right) \\
&= \frac{\partial^{p-3}}{\partial t^{p-3}}\left(\sum_{i=1}^{\infty}\frac{1}{(2i+3)(2i-2)!}\sigma^3(t\sigma)^{2i-2}k^{2i+3}\right) \\
&= \frac{\partial^{p-3}}{\partial t^{p-3}}\left(\sum_{i=0}^{\infty}\frac{1}{(2i+5)(2i)!}\sigma^3(t\sigma)^{2i}k^{2i+5}\right).
\end{aligned}$$

Proceeding this way, it can be shown that,

$$J_p(t) = \begin{cases} \sum_{i=0}^{\infty}\frac{1}{(2i+2d+1)(2i)!}\sigma^{2d-1}(t\sigma)^{2i}k^{2i+2d+1}, & \text{if } p = 2d-1 \\ \sum_{i=1}^{\infty}\frac{1}{(2i+2d+1)(2i-1)!}\sigma^{2d}(t\sigma)^{2i-1}k^{2i+2d+1}, & \text{if } p = 2d \end{cases}$$

Thus, from Eq. (26) we obtain,

$$F_{p+1}(t) = \begin{cases} C\sum_{i=1}^{\infty}\frac{1}{(2i+p)(2i-2)!}\sigma^{p+1}(t\sigma)^{2i-2}k^{2i+p}, & \text{if } p = 2d-1 \\ C\sum_{i=1}^{\infty}\frac{1}{(2i+p+1)(2i-1)!}\sigma^{p+1}(t\sigma)^{2i-1}k^{2i+p+1}, & \text{if } p = 2d \end{cases}$$

At $t = 0$,

$$F_{p+1}(0) = \begin{cases} 2\frac{\phi(k)}{D}\frac{\sigma^{p+1}k^{p+2}}{(p+2)}, & \text{if } p = 2d-1 \\ 0, & \text{if } p = 2d \end{cases} \tag{31}$$

Combining (31) and Lemma 2.5 with $t = 0$, we get:

$$\mu_{p+1} = \begin{cases} p\sigma^2\mu_{p-1} + 2\frac{\phi(k)}{D}\frac{\sigma^{p+1}k^{p+2}}{(p+2)}, & \text{if } p = 2d-1 \\ 0, & \text{if } p = 2d \end{cases}$$

□

Appendix B TABLES

Sample size	μ	σ	k
	MoM	MoM	MoM
20	0.9996 ± 0.0229	0.0774 ± 0.0210	1.7312 ± 1.1972
50	0.9998 ± 0.0144	0.0867 ± 0.0139	1.2220 ± 0.5945
100	1.0001 ± 0.0101	0.0922 ± 0.0092	0.9623 ± 0.3432
1000	1.0000 ± 0.0032	0.0984 ± 0.0029	0.6142 ± 0.1783
5000	1.0001 ± 0.0014	0.0997 ± 0.0014	0.5130 ± 0.1257
10000	1.0000 ± 0.0010	0.0999 ± 0.0011	0.4965 ± 0.1052

Table 1: Performance analysis of the method of moments (MoM) for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 0.1$, and $k = 0.5$.

Sample size	μ	σ	k
	MoM	MoM	MoM
20	1.0004 ± 0.0248	0.0800 ± 0.0224	1.9347 ± 1.3341
50	1.0005 ± 0.0156	0.0879 ± 0.0170	1.5211 ± 0.8959
100	1.0001 ± 0.0108	0.0937 ± 0.0113	1.2375 ± 0.4575
1000	1.0000 ± 0.0035	0.0995 ± 0.0038	1.0111 ± 0.1541
5000	1.0000 ± 0.0015	0.0999 ± 0.0017	1.0030 ± 0.0665
10000	1.0000 ± 0.0011	0.1000 ± 0.0012	1.0013 ± 0.0467

Table 2: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 0.1$, and $k = 1.0$.

Sample size	μ	σ	k
	MoM	MoM	MoM
20	1.0013 ± 0.0497	0.1150 ± 0.0328	2.5952 ± 1.7003
50	1.0004 ± 0.0301	0.1070 ± 0.0301	3.0306 ± 1.7779
100	1.0001 ± 0.0206	0.1016 ± 0.0280	3.2852 ± 1.7903
1000	1.0000 ± 0.0061	0.0955 ± 0.0179	3.3746 ± 1.2251
5000	1.0000 ± 0.0027	0.0989 ± 0.0070	3.0595 ± 0.3267
10000	1.0000 ± 0.0019	0.0994 ± 0.0046	3.0294 ± 0.1939

Table 3: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 0.1$, and $k = 3.0$.

Sample size	μ	σ	k
	MoM	MoM	MoM
20	1.0015 ± 0.2240	0.7713 ± 0.2132	1.7712 ± 1.2748
50	0.9999 ± 0.1427	0.8716 ± 0.1392	1.2161 ± 0.6424
100	1.0016 ± 0.1015	0.9240 ± 0.0913	0.9572 ± 0.3490
1000	0.9998 ± 0.0320	0.9838 ± 0.0295	0.6152 ± 0.1777
5000	1.0000 ± 0.0143	0.9963 ± 0.0143	0.5171 ± 0.1255
10000	0.9999 ± 0.0102	0.9989 ± 0.0106	0.4956 ± 0.1049

Table 4: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 1.0$, and $k = 0.5$.

Sample size	μ MoM	σ MoM	k MoM
20	1.0013 ± 0.2486	0.7969 ± 0.2239	1.9422 ± 1.3136
50	1.0065 ± 0.1539	0.8770 ± 0.1689	1.5325 ± 0.8834
100	1.0018 ± 0.1088	0.9369 ± 0.1126	1.2380 ± 0.4534
1000	1.0002 ± 0.0348	0.9958 ± 0.0379	1.0088 ± 0.1535
5000	0.9998 ± 0.0153	0.9988 ± 0.0171	1.0029 ± 0.0681
10000	1.0000 ± 0.0111	0.9994 ± 0.0121	1.0021 ± 0.0471

Table 5: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 1.0$, and $k = 1.0$.

Sample size	μ MoM	σ MoM	k MoM
20	1.0103 ± 0.5065	1.1422 ± 0.3297	2.6308 ± 1.7389
50	0.9958 ± 0.3012	1.0641 ± 0.2995	3.0486 ± 1.7817
100	0.9977 ± 0.2037	1.0175 ± 0.2816	3.2877 ± 1.7990
1000	1.0004 ± 0.0613	0.9564 ± 0.1771	3.3638 ± 1.2173
5000	1.0001 ± 0.0272	0.9902 ± 0.0672	3.0547 ± 0.2952
10000	1.0005 ± 0.0193	0.9952 ± 0.0460	3.0254 ± 0.1905

Table 6: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 1.0$, and $k = 3.0$.

Sample size	μ	σ	k
	MoM	MoM	MoM
20	0.9904 ± 1.1352	3.8792 ± 1.0553	1.7450 ± 1.2221
50	0.9962 ± 0.7164	4.3631 ± 0.6859	1.2064 ± 0.5706
100	0.9960 ± 0.5061	4.6063 ± 0.4508	0.9687 ± 0.3416
1000	0.9990 ± 0.1573	4.9278 ± 0.1467	0.6060 ± 0.1803
5000	1.0004 ± 0.0709	4.9858 ± 0.0716	0.5096 ± 0.1286
10000	0.9999 ± 0.0505	4.9941 ± 0.0531	0.4954 ± 0.1044

Table 7: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 5.0$, and $k = 0.5$.

Sample size	μ	σ	k
	MoM	MoM	MoM
20	0.9982 ± 1.2551	4.0027 ± 1.1303	1.9508 ± 1.3655
50	1.0167 ± 0.7744	4.4116 ± 0.8317	1.5112 ± 0.8648
100	0.9962 ± 0.5444	4.6807 ± 0.5573	1.2387 ± 0.4347
1000	0.9980 ± 0.1743	4.9801 ± 0.1895	1.0097 ± 0.1514
5000	1.0011 ± 0.0773	4.9945 ± 0.0838	1.0023 ± 0.0661
10000	0.9989 ± 0.0559	4.9980 ± 0.0606	1.0012 ± 0.0473

Table 8: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 5.0$, and $k = 1.0$.

Sample size	μ	σ	k
	MoM	MoM	MoM
20	0.9317 ± 2.4752	5.6880 ± 1.6528	2.6572 ± 1.7507
50	0.9953 ± 1.5184	5.3792 ± 1.5081	3.0103 ± 1.7866
100	1.0078 ± 1.0237	5.1089 ± 1.3623	3.2308 ± 1.7037
1000	1.0050 ± 0.3007	4.7678 ± 0.8955	3.3770 ± 1.2140
5000	0.9982 ± 0.1362	4.9451 ± 0.3535	3.0610 ± 0.3204
10000	1.0000 ± 0.0959	4.9786 ± 0.2310	3.0238 ± 0.1929

Table 9: Performance analysis of the MoM for the estimation of μ, σ, and k from simulation experiment with $\mu = 1.0$, $\sigma = 5.0$, and $k = 3.0$.