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# Newsvendor Problem with Unknown Demand Distribution and Non-Linear Underage and Overage Cost

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# Newsvendor problem with unknown demand distribution and non-linear underage and overage cost

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**Abstract**

This paper attempts to develop a newsvendor model to calculate the optimal order quantity under non-linear overage and underage cost with unknown demand distribution. In this paper, we consider a single-period newsvendor model which is investigated by assuming that the overage and underage cost function depends on level of inventory. We have derived criterion for existence of real root for quadratic and cubic loss functions. We also inspect the estimation problem of the optimal order quantity when the demand distribution is unknown. Finally we have done some simulations to see the accuracy and efficiency of the estimated order quantities.

## 1. Introduction

The newsvendor problem is one of the classical problems in the literature on inventory management Silver et.al. (1998). Newsvendor problem relies on offsetting the underage and overage cost in order to obtain optimal order quantity. In the standard newsvendor problem, underage and overage costs are assumed to be proportional to the loss amount, i.e linear in the order quantity. However, there may be situations where the loss is more severe than the linear usual linear loss. On the other hand, stochastic inventory models assume the demand to be a random variable with known probability distributions. In reality, the distribution is often unknown, at least in the sense of parameter values. In this paper, we consider the stochastic newsboy problem with non-linear losses and unknown demand distributions.

Classical newsvendor problem is described as follows. Consider a newsvendor selling a short lived commodity procured from a single supplier. Further assume the demand to be a random variable. Newsvendor takes one time decision on how much quantity she should order from supplier. The newsvendor faces overage cost if the demand is lower than the inventory, as a penalty for ordering too much. Similarly, shortage cost is faced if the demand is higher than the inventory, as a penalty for ordering too less. The shortage (overage) cost is computed in terms of currency units as the product of per unit shortage (overage) cost multiplied with amount of loss (i.e difference between demand and inventory). To determine the optimal order quantity, the newsvendor has to minimize the total cost or maximize the total profit. Hadley and Whitin (1963) studied a newsvendor model with multiple products. Veinott (1965) generalized the newsvendor model for multiple time periods. We refer to a recent literature review by Qin et.al.(2013) for other extensions of newsvendor problem adding dimensions such as marketing efforts, risk taking nature of buyer etc.

Newsvendor problem with non-linear losses has attracted researchers in recent years. For example, Parlar & Rempala (1992) considered the periodic review inventory problem and derived the solution for a quadratic loss function for both shortage and overage. Gerchak & Wang (1996) proposed a newsvendor model using power type loss function for asset allocation. Their work assumes linear overage but power type underage loss. However, the work details only the quadratic loss. Optimal order quantity derived from a power type loss function with unknown demand distribution is a problem that remains unsolved to the best of the knowledge of the authors.

In this paper we consider newsvendor problem with – (1) power type loss function with degree more or equal to two and (2) unknown demand distribution over a non-negative support. We first show whether the cost minimization is equivalent to profit maximization or not. Next we derive analytically the optimal order quantity for uniform distribution over a non-negative interval for general power function. Next, we derive the maximum likelihood estimator (MLE) of the optimal order quantity and study its statistical properties through simulation.

The rest of the paper is organized as follows. Next section describes the main results. In section 3 we provide numerical examples and simulation study. Section 4 is the concluding section with discussion on the results and possible extensions.

## 2. Main results

**<u>Problem Description</u>**

The aim of this paper is to find out the optimal quantity taking non-linear and symmetric overage and underage cost function.

Let us consider newsvendor problem for single period and single product under stochastic demand. The newsvendor orders the entire inventory once at the beginning of the period. We aim to find optimal order quantity for the newsvendor. The following notations will be used throughout the paper.

**<u>Model Notations</u>-**

**X**- Random demand

$\boldsymbol{f_\theta(x)}$**-** Probability Density function of X (pdf)

$\boldsymbol{F_\theta(x)}$**-** Cumulative density function (CDF)

$\theta$: unknown parameter (may be vector valued) $\in \Theta$

**c**- Purchasing cost

**p**- Selling price

$\mathbf{c_s}$ - Shortage cost

$\mathbf{c_0}$- Overage cost

**Q-** Order Quantity

We assume that sales opportunity will be lost if the product is not available (i.e. no backlog cost). We also assume fixed purchasing and sales price and absence of any influence of the factors like marketing efforts, promotions, discounts etc. The support of the random demand, say $\mathfrak{S}$, is assumed to be non-negative and the CDF $F_\theta(x)$, is assumed to be continuous, differentiable and strictly increasing over $\mathfrak{S}$.

With realized demand *x* and order quantity Q, cost function can be expressed as-

$$\mathrm{C(Q, x)} = \begin{cases} c_s(X-Q)^m, & if\ X > Q \\ c_o(Q-X)^m, & if\ X \le Q \end{cases}$$

so that the expected cost is given by

$$\chi_{\mathrm{m}}(Q) = E[C_m(Q,X)] = \int_{\inf(\mathfrak{S})}^{Q} c_0\,(Q - x)^m f(x)dx + \int_{Q}^{\sup(\mathfrak{S})} c_s\,(x - Q)^m f(x)dx$$

Hence the FOC can be easily shown to be

$$\frac{\partial \chi_m(Q)}{\partial Q} = \int_{\inf(\mathfrak{S})}^{Q} \mathrm{mc}_0\,(Q - x)^{m-1} f(x)\,\mathrm{dx} - \int_{Q}^{\sup(\mathfrak{S})} \mathrm{mc}_s\,(x - Q)^{m-1} f(x)\,\mathrm{dx} = 0$$

In particular, the solution for *m=1* is given by $Q^* = F^{-1}\left(\frac{\mathrm{cs}}{c_0+c_s}\right)$, which coincides with the solution for standard newsvendor problem. For *m=2* onwards it is not possible to find reduced form expression without the knowledge of $f_\theta(x)$. Hence forward we shall assume a non-informative distribution, viz. uniform over an interval *(a,b)*. In that case FOC reduces to the following quadratic equation in Q:

$$\mathrm{Q}^2(c_o - c_s) + 2Q(\mathrm{b}\,\mathrm{c}_s - ac_o) - c_s(b^2 - a^2) = 0$$

and the roots of the quadratic equation is given by

$$Q^* = \frac{-(\mathrm{bc}_s - ac_o) \pm \sqrt{(\mathrm{bc}_s - ac_o)^2 - (b^2 - a^2)c_s(\mathrm{c}_o - c_s)}}{c_o - c_s}$$

Particularly, *a=0* corresponds to a situation where there is no sure sell (like pre-booking). Even if there is, then one may think of Q as the additional variable quantity that needs to be determined optimally. In that case the optimal order quantity will be real, iff $c_s \geq \frac{c_o}{2}$.

For *m=3*, the FOC reduces to the following cubic equation.

$$Q^3 - 3\,Q^2(\lambda b + (1-\lambda)a) + 3Q(\lambda b^2 + (1-\lambda)a^2) - (\lambda b^3 + (1-\lambda)a^3) = 0$$

where $\lambda = \frac{c_s}{c_o+c_s} \in (0,1)$. In particular, if *a=0*, then the determinant of the cubic equation becomes $-27\lambda^2 b^6(\lambda - 1)^2 < 0$ and hence there will be only one real solution leading to a unique optimal order quantity. For higher powers of the loss function, numerical methods would be required to solve the respective FOC equations.

Now let us suppose that the parameters of the uniform distribution, i.e *a* and *b*, are unknown and a random sample of size *n* on demand (measured by exact sales quantity) is available. Let us denote the sample by $X_1, X_2 \ldots X_n$, which are identically and independently distributed (iid) as *U(a,b)*. Since the solutions are single valued functions of *a* and *b*, the MLE of the optimal order quantity would be obtained by replacing the parameters with their MLEs. For uniform distribution, $\hat{a}_{MLE} = X_{(1)}$ and $\hat{b}_{MLE} = X_{(n)}$, where $X_{(i)}$ indicates the $i^{th}$ order statistic, $i$=1, 2 … *n*. Thus it is straight forward to obtain the MLE of the optimal order quantity from estimated FOCs. In the following section we present simulation study to find the estimated optimal order quantity.

## 3. Simulation

In this section we consider *Unif(10,20)* to observe the performance of the estimated optimal order quantity. Further we assume, $c_o$ = 20 and $c_s$ =25. We compute the true $Q$* as detailed above.

In the simulation we draw samples of size n (=10, 20, …, 100). For a given sample of size *n* the estimated optimal order quantity ($Q^*_{MLE}$) is determined and this procedure is repeated 10000 times. We compute the bias in the estimated optimal order quantity by the average of ($Q^*_{MLE} - Q^*$) over these 10000 repetitions. In the following table we show the bias and mean square errors (MSE) corresponding to different sample sizes.

| | m=1, Q*=15.56 | | m=3, Q*=15.18 | | m=5, Q*=15.11 | |
|---|---|---|---|---|---|---|
| *n* | Average Bias | MSE | Average Bias | MSE | Average Bias | MSE |
| 10 | 0.4658 | 0.1874 | 0.4593 | 0.1762 | 0.4584 | 0.1750 |
| 20 | 0.2478 | 0.0625 | 0.2427 | 0.0587 | 0.2421 | 0.0582 |
| 30 | 0.1595 | 0.0213 | 0.1569 | 0.0202 | 0.1566 | 0.0201 |
| 40 | 0.1284 | 0.0154 | 0.1269 | 0.0148 | 0.1269 | 0.0148 |
| 50 | 0.1012 | 0.0091 | 0.1004 | 0.0088 | 0.1004 | 0.0088 |
| 60 | 0.0828 | 0.0065 | 0.0813 | 0.0061 | 0.0811 | 0.0061 |
| 70 | 0.0738 | 0.0051 | 0.0727 | 0.0048 | 0.0726 | 0.0048 |
| 80 | 0.0635 | 0.0042 | 0.0623 | 0.0040 | 0.0621 | 0.0040 |
| 90 | 0.0585 | 0.0039 | 0.0577 | 0.0036 | 0.0577 | 0.0036 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 100 | 0.0486 | 0.0024 | 0.0484 | 0.0023 | 0.0484 | 0.0023 |

*Table 1: Bias and MSE of estimated optimal order quantity with simulation size 10000 for m=1,3 & 5*

We observe that the optimal order quantity decreases with the power of the loss function. This result is intuitive as higher the degree, more severe is the loss. In case of severe loss one would expect to order a little less. However, if the degree of loss differs, then the optimal order quantity would correspond to the loss that is more severe. Further, the bias as well as MSE decreases with *n* for each *m.* However, the amount of bias becomes nearly constant for m=3 and 5. Thus the accuracy and efficiency of the estimated optimal order quantity seems to be independent of the severity of loss. So, we expect this method of estimation to be robust across all level of severity, i.e different values of *m*.

## 4. Discussion

In this paper we have described inventory problem with severe losses, incorporated through power type loss functions with different degrees. We have considered stochastic demand with unknown distribution. We have derived the conditions for existence of optimal order quantity with quadratic loss and *unif(a,b)* distribution. We have also shown that cubic losses will result in unique optimal order quantity. Numerical example in this work reveals that higher the severity of loss, lesser the order quantity should be. Simulation study reveals that the estimated optimal order quantity becomes more accurate and efficient as the sample size increases. Since the bias and MSE does not change much with respect to the degree of the loss function ($m > 2$), the MLE indication to be quite a robust estimator.

As scope of future work, it would be interesting to compare the performances under other popular demand distributions available in the literature. Also the asymptotic distribution of the estimated optimal order quantity with respect *m* (i.e. increasing loss severity) would be explored empirically.

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