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On Linnik distribution, Linnik Levy Process and Generalized Linnik Levy process

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On Linnik distribution, Linnik Lévy Process and Generalized Linnik Lévy process

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Abstract. In the literature, the Linnik distribution together with the Mittag-Leffler one are presented as physically relevant examples of geometric stable distributions. The geometric stable distributions are especially useful in modeling leptokurtic data with heavy-tailed behavior. They have found many interesting applications, including physical phenomenon and finance. In this paper, we define the Linnik Lévy processes (LLP) through the subordination of the stable Lévy motion by the gamma process. We discuss main properties of LLP like probability density function and corresponding Lévy measure. We consider also the governing fractional-type Fokker-Planck equation. Further, a generalization of the introduced process is also discussed.

Keywords: Linnik distribution, subordinated stochastic processes, Lévy density.

1. Introduction

In the literature the Linnik distribution together with the Mittag-Leffler one are presented as the most known examples of geometric stable distributions. The geometric stable distributions belong to the family of leptokurtic probability distributions. One can see the link between stable (called also Lévy stable or $\alpha-$stable) distributions and the geometric ones. One of the most important property of stable distribution is as follows: if $X_1, X_2, \ldots, X_n$ are independent, identically stable distributed, then its sum $Y = a_n(X_1 + X_2 + \ldots + X_n) + b_n$ has the same distribution as X_is for some constants a_n and b_n. For the geometric stable distribution we have the similar property: if $X_1, X_2, \ldots, X_n$ are independent, identically geometric stable distributed and N_p is a random variable independent on X_is having geometric distribution with parameter p, then the sum $Y = a_{N_p}(X_1+X_2+\ldots+X_{N_p})+b_{N_p}$ approaches the distribution of X_is when p tends to zero, [1]. The geometric stable distributions are considered as an alternative for the stable ones, which have the features making them suitable for many real data. One of the classical application of stable-based processes are financial markets. In most of the cases the financial data exhibit so-called heavy-tailed behavior therefore the stable-based processes seem to be more appropriate than the Gaussian-based systems. As it was shown in [2] the geometric stable distribution was selected as the best to the Yen exchange rate description. Since the geometric stable distributions arise as the sum of random variables, they naturally arise in many real problems [3, 4] and are especially important in modeling heavy-tailed data, when the considered process can be analyzed as a sum of independent observations when their number is also random. This effect is often observable in financial phenomenon [1].

The Linnik distribution (called also a Laplace distribution) is a special case of the geometric distributions. It was introduced by Ju.V. Linnik in [5] and is widely studied by many authors, see e.g [6–8]. The distribution function of Linnik random variable is not given in the closed form and hence it is generally defined in term of characteristic function (c.f.). If X is the Linnik-distributed random variable then its c.f. is defined by

$$\phi_X(u) = \mathbb{E}(e^{iuX}) = \frac{1}{(1+|u|^\alpha)},\ 0 < \alpha < 2. \tag{1}$$

The presented above c.f. can be viewed as a generalization of the well-known Laplace (double exponential) c.f. $\phi(u) = 1/(1 + u^2)$ for the case $\alpha = 2$, see, e.g. [9]. Since Linnik distributions are infinitely divisible [10] one can define a continuous time Lévy process for these distributions. In the literature Linnik Lévy processes (LLPs) are also called geometric stable processes (GSP), see e.g. [11, 12]. Per our knowledge, the explicit expressions for the probability density function (PDF), Lévy measure, governed fractional Fokker-Planck equation as well as the tail behavior for the LLP are not available in the literature. Our paper fills that gap, generalizes and complements the results available on Linnik distribution and processes in different directions.

On one hand, the LLP is a process of stationary independent increments having Linnik distribution. On the other hand the LLP can be represented as the subordination of the stable Lévy motion by the gamma process, called gamma subordinator. In general, the idea of subordination was introduced in 1949 by Bohner [13] and is based by changing the time of a stochastic process by some other process which is generally a non-decreasing Lévy process, called subordinator. The theory of subordinated processes is explored in details in [14]. The subordinated processes were studied in many areas of interest, for example in finance [15–20], physics [21–24], ecology [25], hydrology [26] and biology [27]. In the literature one can find different examples of subordinated processes. One of the classical one is the variance gamma process known also as the Laplace motion [28, 29]. The variance gamma process was applied in different fields however the classical applications are financial data. It arises as the Brownian motion with drift subordinated by the gamma process. Thus the LLP can be considered as the extension of the popular subordinated process and thus can find even more applications than the classical process. In this paper we demonstrate also how to generalize the LLP to the more general Lévy processes which can be used to description of heavy-tailed behavior. The rest of the paper is organized as follows: in section 2 we introduce the LLP process as the subordinated stable Lévy motion delayed by gamma process. In the following subsections we present the main properties of the introduced process, like probability density function and Lévy measure, governing fractional-type Fokker-Planck equation, tail behavior and fractional moments. In section 3 we generalize the LLP process through replacement of the gamma process by general Lévy subordinator. We present also main properties of the generalized process. Last section concludes the paper.

2. Linnik Lévy Process (LLP)

Before we define the Linnik Lévy process we remind the definition and main properties of the Stable Lévy motion and gamma Lévy process, two processes used to the LLP construction.

2.1. Stable Lévy motion

A stable distribution is also called an α-stable or a Lévy stable. Probability density function (PDF) of stable distribution does not possess closed form except for three cases (Gaussian, Cauchy and Lévy). Therefore it is more convenient to express stable PDF in term of its characteristic function (or Fourier transform). A random variable X is said to follow stable distribution with parameters $\alpha, \tilde{\beta}, \mu$ and σ if its characteristic function $\phi(u)$ satisfies [30]

$$\log\phi(u) = \log\mathbb{E}\left(e^{iXu}\right) = \begin{cases} -\sigma^{\alpha}|u|^{\alpha}[1 - i\tilde{\beta}\mathrm{sign}(u)\tan(\frac{u\alpha}{2})] + i\mu u, & \text{if } \alpha \neq 1 \\ -\sigma|u|[1 + i\tilde{\beta}\mathrm{sign}(u)\frac{2}{\pi}\ln|u|] + i\mu u, & \text{if } \alpha = 1, \end{cases} \tag{1}$$

where $\alpha \in (0, 2]$ is the stability index, $\tilde{\beta} \in [-1, 1]$ is the skewness, $\mu \in \mathbb{R}$ is the location (or shift) and $\sigma > 0$ is the scale. For $\alpha = 2$ the stable random variable is Gaussian. In this paper we consider the special symmetric case, namely $\tilde{\beta} = \mu = 0$.

A stable Lévy motion $S(t)$, $t \geq 0$ has stationary independent increments with $S(t)-S(s)$ $(t > s)$ having stable distribution with parameters α, $\sigma = (|t-s|)^{1/\alpha}$ and $\tilde{\beta} = \mu = 0$. Further, the characteristic function of $S(t)$ [30] is given by

$$\mathbb{E}(e^{iuS(t)}) = e^{-t|u|^\alpha}, \ \alpha \in (0,2], t > 0, u \in \mathbb{R}. \tag{2}$$

For large x right tail of the stable Lévy motion for $0 < \alpha < 2$ behaves [30]

$$\mathbb{P}(S(t) > x) \sim C_\alpha t x^{-\alpha}, \tag{3}$$

where

$$C_\alpha = \begin{cases} \frac{(1-\alpha)}{2\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}, & \text{if } \alpha \neq 1 \\ \frac{1}{\pi}, & \text{if } \alpha = 1. \end{cases} \tag{4}$$

It is worth to highlight some crucial properties of the process $S(t)$. One of them is self-similarity property, which means all finite-dimensional distributions of $\{S(at), t > 0\}$ agree with those of $\{a^{1/\alpha}S(t), t > 0\}$. Moreover, distribution of the $S(t)$ is self-decomposable, which means that its Lévy measure has a certain form

$$\pi_S(dx) = \frac{k(x)}{|x|}$$

with the function $k(x)$ which is decreasing on $(0,\infty)$ and increasing on $(-\infty, 0)$. More details one can find in [31]. Using self-similarity property of stable Lévy motion and [32], p. 583, one can show the marginal PDF of the process $S(t)$ in series form is given by

$$f(x,t) = \begin{cases} \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(k\alpha+1)}{k!}\frac{t^k}{x^{k\alpha+1}}\sin\left(\frac{\pi\alpha k}{2}\right), & \text{if } x \in \mathbb{R}, \ 0 < \alpha < 1 \\ \dfrac{1}{\pi}\displaystyle\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(k/\alpha+1)}{k!}t^{-k/\alpha}x^{k-1}\sin\left(\frac{\pi k}{2}\right), & \text{if } x \in \mathbb{R}, \ 1 < \alpha < 2. \end{cases} \tag{5}$$

For $\alpha = 2$, the stable Lévy motion reduces to standard Brownian motion while for $\alpha = 1$ it is known as Cauchy process. Thus we have

$$f(x,t) = \begin{cases} \frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}, & x \in \mathbb{R}, \ \alpha = 2 \\ \frac{t}{\pi}\frac{1}{x^2+t^2}, & x \in \mathbb{R}, \ \alpha = 1. \end{cases} \tag{6}$$

2.2. Gamma process

The gamma Lévy process $G(t)$, $t \geq 0$ (called simple gamma process) has independent increments of gamma distribution, i.e. $G(t) - G(s) \sim \text{Gamma}(\lambda, \beta(t-s))$, where a random variable Y is said to have $\text{Gamma}(\lambda,\beta)$ distribution if its PDF has the form

$$f_Y(y) = \frac{\lambda^\beta}{\Gamma(\beta)}y^{\beta-1}e^{-\lambda y}, \ y > 0.$$

Further, the Laplace transform (LT) of $G(t)$ is given by

$$\mathbb{E}\left(e^{-uG(t)}\right) = \left(\frac{\lambda}{\lambda + u}\right)^{\beta t}. \tag{7}$$

For $q > 0$, the q-th order moment for $G(t)$ is

$$\mathbb{E}(G(t)^q) = \frac{\Gamma(\beta t + q)}{\Gamma(\beta t)}\lambda^{-q}. \tag{8}$$

The Lévy density for gamma process is, [31]

$$\pi_G(x) = \frac{\beta}{x}e^{-\lambda x}. \tag{9}$$

2.3. Main properties of LLP

In this subsection first we define Linnik Lévy process by subordinating a stable Lévy motion with the gamma process. Both processes and their properties were introduced above. Note that, the representation of LLP through the subordination scenario is very helpful in deriving the properties of the LLP (in general) and Linnik distribution (in particular). This way we generalize and complements the results obtained on Linnik distribution in different directions. After presenting the definition we give main properties of introduced process.
The LLP denoted by $X(t)$ is defined by

$$X(t) := S(G(t)), \quad t \geq 0, \tag{10}$$

where $S(t)$ and $G(t)$ are independent stable and gamma Lévy processes, respectively. Using a standard conditioning argument with (2) and (7), it follows

$$\mathbb{E}(e^{iuX(t)}) = \mathbb{E}\left(\mathbb{E}\left(e^{iuS(G(t))}|G(t)\right)\right) = \mathbb{E}\left(e^{-G(t)|u|^\alpha}\right) = \left(\frac{\lambda}{\lambda + |u|^\alpha}\right)^{\beta t}. \tag{11}$$

For $\lambda = \beta = t = 1$, the characteristic function of $X(t)$ given in (11) reduces to the characteristic function of Linnik distribution given in (1).

Probability density function and Lévy measure of LLP

Let us denote $f(x,t)$ as the marginal PDF of stable Lévy motion $S(t)$. For $\beta = 1$, LLP is a stochastically self-similar process with self-similarity parameter $1/\alpha$ with respect to the family of negative binomial Lévy processes. This follows using the fact that the LLP is obtained by subordinating the stable Lévy motion, which is self-similar with index $1/\alpha$, with a gamma process which is stochastically self-similar with respect to the family of negative binomial Lévy processes (see Proposition 4.2 in [33]). The PDF of process $X(t)$ can be represented by

$$h(x,t) = \int_0^\infty f(x,y)g(y,t)dy, \; x \in \mathbb{R}, \; t \geq 0.$$

Moreover, the Lévy measure π_X for $X(t)$ can be written as [34]

$$\pi_X(x) = \int_0^\infty f(x,y)\pi_G(y)dy. \tag{12}$$

Using Gaussian PDF (i.e. stable with $\alpha = 2$), the Lévy measure for $X(t)$ for $\alpha = 2$ can be calculated explicitly as follows

$$\begin{aligned}
\pi_X(x) &= \int_0^\infty f(x,y)\pi_G(y)dy = \int_0^\infty \frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} \frac{\beta}{y} e^{-\lambda y} dy \\
&= \frac{\beta}{\sqrt{2\pi}} \int_0^\infty y^{-3/2} e^{-\frac{1}{2}(\frac{x^2}{y}+2\lambda y)} dy \quad (\text{substitute } y = \tfrac{x}{\sqrt{2\lambda}} z) \\
&= \frac{\beta}{\sqrt{2\pi}} \left(\frac{x}{\sqrt{2\lambda}}\right)^{-1/2} \int_0^\infty z^{-3/2} e^{-\frac{1}{2}x\sqrt{2\lambda}(\frac{1}{z}+z)} dz \\
&= \frac{\beta}{\sqrt{2\pi}} \left(\frac{x}{\sqrt{2\lambda}}\right)^{-1/2} K_{-1/2}(x\sqrt{2\lambda}) = \frac{\beta}{2x} e^{-x\sqrt{2\lambda}},
\end{aligned}$$

where $K_\nu(\omega)$ is the modified Bessel function of third kind with index ν, defined by [35]

$$K_\nu(\omega) = \frac{1}{2}\int_0^\infty x^{\nu-1} e^{-\frac{1}{2}\omega(x+x^{-1})} dx, \ \omega > 0. \tag{13}$$

Further,

$$K_{1/2}(\omega) = K_{-1/2}(\omega) = \sqrt{\frac{\pi}{2\omega}} e^{-\omega}.$$

For $0 < \alpha < 1$, using (5) and (12), it follows

$$\pi_X(x) = \frac{\beta}{\pi x} \sum_{k=1}^\infty (-1)^{k+1} \frac{\Gamma(1+k\alpha)}{k x^{k\alpha} \lambda^k} \sin\left(\frac{\pi\alpha k}{2}\right). \tag{14}$$

For $\alpha = 1$,

$$\pi_X(x) = \frac{\beta}{\pi} \int_0^\infty \frac{e^{-\lambda y}}{x^2+y^2} dy \sim \frac{\beta}{\pi \lambda x^2}, \text{ as } x \to \infty.$$

For $1 < \alpha < 2$, the Lévy density seems difficult to write in a closed form and can be represented in the integral form given in (12). However, an asymptotic form for $\pi_X(x)$ for $\alpha \in (1,2)$ can be found by using (3), (9) and (12). It is given by

$$\pi_X(x) \sim \frac{\beta}{\lambda\pi} \Gamma(1+\alpha) \sin(\pi\alpha/2) \frac{1}{x^{1+\alpha}}, \quad \text{as} \quad x \to \infty. \tag{15}$$

For asymptotic behavior of Lévy density as $x \to 0^+$, see [11].

Proposition 2.1. *For $\alpha \in (0,2)$ the PDF $h(x,t)$ of $X(t)$ can be represented as*

$$h(x,t) = \frac{\lambda^{\beta t}}{\pi} \mathrm{Re} \int_0^\infty \frac{i e^{-sx}}{(\lambda + s^\alpha e^{i\pi\alpha/2})^{\beta t}} ds, \ x > 0. \tag{16}$$

Proof. By Fourier inversion formula, we have

$$h(x,t)=\frac{\lambda^{\beta t}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iux}}{(\lambda+|u|^{\alpha})^{\beta t}}du=\frac{\lambda^{\beta t}}{\pi}\int_{0}^{\infty}\frac{\cos ux}{(\lambda+u^{\alpha})^{\beta t}}du=\frac{\lambda^{\beta t}}{\pi}\mathrm{Re}\int_{0}^{\infty}\frac{e^{iux}}{(\lambda+u^{\alpha})^{\beta t}}du. \tag{17}$$

Now let us consider the complex region $S_R=\{u=r+is:|u|<R,\ r>0,\ s>0\}$. S_R is the portion of the disc of radius R in the positive quadrant. The function $(\lambda+u^{\alpha})^{\beta t}$ is analytic in S_R. By Cauchy integral formula, we have

$$\oint_{\partial S_R}\frac{e^{iux}}{(\lambda+u^{\alpha})^{\beta t}}du=0.$$

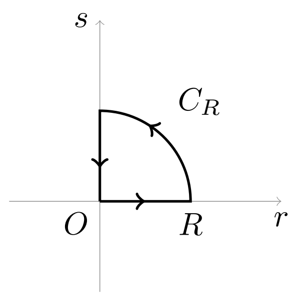

Figure 1. Contour C_R.

Let $C_R=\{u=r+is:|u|=R,r\geq 0,s\geq 0\}$, then we have

$$\int_0^R\frac{e^{irx}}{(\lambda+r^{\alpha})^{\beta t}}dr+\int_{C_R}\frac{e^{iux}}{(\lambda+u^{\alpha})^{\beta t}}du-\int_0^R\frac{e^{-sx}}{(\lambda+s^{\alpha}e^{i\pi\alpha/2})^{\beta t}}ids=0.$$

The integral along C_R tends to 0 as $R\to\infty$ and hence

$$\int_0^{\infty}\frac{e^{irx}}{(\lambda+r^{\alpha})^{\beta t}}dr=\int_0^{\infty}\frac{e^{-sx}}{(\lambda+s^{\alpha}e^{i\pi\alpha/2})^{\beta t}}ids.$$

Using (17) the result follows. □

Remark 2.1. *For $\lambda=\beta=t=1$ the PDF $h(x,t)$ takes the form*

$$\begin{aligned}h(x,1)&=\frac{1}{\pi}\mathrm{Re}\int_0^{\infty}\frac{ie^{-sx}}{(1+s^{\alpha}e^{i\pi\alpha/2})}ds=\frac{1}{\pi}\mathrm{Re}\int_0^{\infty}\frac{ie^{-sx}(1+s^{\alpha}e^{-i\pi\alpha/2})}{|1+s^{\alpha}e^{i\pi\alpha/2}|^2}ds\\&=\frac{\sin(\pi\alpha/2)}{\pi}\int_0^{\infty}\frac{e^{-sx}s^{\alpha}}{|1+s^{\alpha}e^{i\pi\alpha/2}|^2}ds,\end{aligned}$$

which is the Linnik PDF, see e.g. [36].

Remark 2.2. *For $\alpha=2$, the PDF $h(x,t)$ represents symmetric variance gamma PDF and is given by*

$$h(x,t)=\frac{\lambda^{\beta t}}{\sqrt{2\pi}\Gamma(\beta t)}\left(\frac{x}{\sqrt{2\lambda}}\right)^{\beta t-1/2}K_{\beta t+1/2}(x\sqrt{2\lambda}),\ x\in\mathbb{R}. \tag{18}$$

Next we discuss the asymptotic behavior of the marginal PDF of $X(t)$. It is required to use the following theorem given in [37].

Theorem 2.1. *[37] If the characteristic function $\phi(u)$ for a random variable X is absolutely integrable and can be decomposed into the form*

$$\phi(u) = \phi_1(u) + \phi_2(u) + \phi_3(u),$$

where

$$\phi_1(u) = e^{i\eta u} \sum_{m=0}^{M-1} A_m (iu)^m, \quad \phi_2(u) = e^{i\eta u} |u|^p \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \sum_{l=0}^{L(j)} B_{jkl} |u|^{rj} (i\mathrm{sgn}\,(u))^k (\ln |u|)^l, \ p \geq M, \ r > 0,$$

and $\phi_3^{(j)}(u)$ is absolutely integrable for $j = 0, 1, 2, \cdots, N$ and N is the smallest integer $\geq p + jr + 1$, also $\phi_3^{(j)}(u) \to 0$ as $u \to \pm\infty$ for $j = 0, 1, 2, \cdots, N$, then the PDF $f_X(s)$ has the following asymptotic expansion as $|x| \to \infty$

$$f_X(x) \sim \frac{1}{\pi |x - \eta|^{p+1}} \sum_{j=0}^{J} \sum_{k=0}^{K(j)} \Bigg[\sum_{l=0}^{L(j)} B_{jkl} \frac{\partial^l}{\partial z^l} \Gamma(z + p + 1) |y|^{-z} \tag{19}$$

$$\cdot \frac{1}{2} \left\{ i^k e^{-\frac{1}{2} i\pi \mathrm{sgn}\,(y)(z+p+1)} + (-1)^k e^{\frac{1}{2} i\pi \mathrm{sgn}\,(y)(z+p+1)} \right\} \Bigg]_{z=jr, y=x-\eta} + \mathrm{O}(|x|^{-N}). \tag{20}$$

Proposition 2.2. *The PDF $h(x,t)$ of LLP $X(t)$ has following asymptotic expansion as $|x| \to \infty$*

$$h(x,t) = \frac{1}{\pi |x|^{\alpha+1}} \sum_{j=0}^{J} (-1)^{j+1} \binom{\beta t + j}{j+1} \frac{\Gamma(j\alpha + \alpha + 1)}{\lambda^{j+1}} |x|^{-j\alpha} \cos\left(\frac{\pi}{2} \mathrm{sgn}\,(x)(j\alpha + \alpha + 1) \right) + \mathrm{O}(|x|^{-N}), \tag{21}$$

where N is smallest integer $\geq (J+1)\alpha + 1$.

Proof. Note that the characteristic function of LLP $X(t)$ can be decomposed as

$$\begin{aligned}
\phi_X(u;t) &= \left(1 + \frac{|u|^\alpha}{\lambda}\right)^{-\beta t} = \sum_{j=0}^{\infty} (-1)^j \binom{\beta t + j - 1}{j} \left(\frac{|u|^\alpha}{\lambda}\right)^j, \ |u| < \lambda^{1/\alpha} \\
&= 1 + \sum_{j=1}^{J} (-1)^j \binom{\beta t + j - 1}{j} \left(\frac{|u|^\alpha}{\lambda}\right)^j + \sum_{j=J+1}^{\infty} (-1)^j \binom{\beta t + j - 1}{j} \left(\frac{|u|^\alpha}{\lambda}\right)^j \\
&= \underbrace{1}_{\phi_1(u)} + \underbrace{|u|^\alpha \sum_{j=0}^{J} (-1)^{j+1} \binom{\beta t + j}{j+1} \frac{|u|^{\alpha j}}{\lambda^{j+1}}}_{\phi_2(u)} + \underbrace{\sum_{j=J+1}^{\infty} (-1)^j \binom{\beta t + j - 1}{j} \left(\frac{|u|^\alpha}{\lambda}\right)^j}_{\phi_3(u)}.
\end{aligned}$$

Using, Theorem 2.1, it follows that $\eta = 0, p = \alpha, r = \alpha, K(j) = 0, L(j) = 0$. Thus as $|x| \to \infty$ we obtain

$$h(x,t) = \frac{1}{\pi |x|^{\alpha+1}} \sum_{j=0}^{J} (-1)^{j+1} \binom{\beta t + j}{j+1} \frac{\Gamma(j\alpha + \alpha + 1)}{\lambda^{j+1}} |x|^{-j\alpha} \cdot \frac{1}{2} \left[e^{-i\frac{\pi}{2}\operatorname{sgn}(x)(j\alpha+\alpha+1)} + e^{-i\frac{\pi}{2}\operatorname{sgn}(x)(j\alpha+\alpha+1)} \right] + \mathrm{O}(|x|^{-N}),$$

which concludes the proof. □

Governing fractional-type Fokker-Planck equation

In this part we discuss shortly the governing fractional-type Fokker-Planck equation (FFPE) for the one dimensional density $h(x,t)$ of the process $X(t)$. Recall the definition from [38] of the so-called shift operator $e^{-p\partial_t}, p \in \mathbb{R}$ which is given by

$$e^{-p\partial_t} f(t) = \sum_{j=0}^{\infty} \frac{(-p\partial_t)^j}{j!} f(t) = f(t-p), \; p \in \mathbb{R}.$$

We denote operator $(-\Delta)^a, a \in (0,1)$ as the fractional Laplacian, which gives the standard Laplacian when $a = 1$ [39]. The fractional Laplacian of order a ($(-\Delta)^a$) can be defined on functions $g : \mathbb{R} \to \mathbb{R}$ as a Fourier multiplier given by the formula

$$\mathcal{F}((-\Delta)^a g))(u) = |u|^{2a} \mathcal{F}(g)(u).$$

Proposition 2.3. *The one dimensional PDF of LLP satisfies the following FFPE*

$$\lambda(1 - e^{-\frac{1}{\beta}\partial_t}) h(x,t) = -(\Delta_x)^{\alpha/2} h(x,t), \; h(x,0) = \delta_0(x). \tag{22}$$

Proof. Taking Fourier transform with respect to the space variable denoted by $\mathcal{F}_x$ in left hand side of the equation (22) we obtain

$$\begin{aligned} \mathcal{F}_x \left(\lambda(1 - e^{-\frac{1}{\beta}\partial_t}) h(x,t) \right) &= \lambda(1 - e^{-\frac{1}{\beta}\partial_t}) \left(\frac{\lambda}{\lambda + |u|^\alpha} \right)^{\beta t} \\ &= \lambda \left(\frac{\lambda}{\lambda + |u|^\alpha} \right)^{\beta t} - \lambda \left(\frac{\lambda}{\lambda + |u|^\alpha} \right)^{\beta t - 1} = -|u|^\alpha \mathcal{F}_x(h(x,t)). \end{aligned}$$

By inverting the Fourier transform one can obtain the desired result. □

Tail behavior of LLP

Proposition 2.4. *The tail of the distribution for $X(t)$ has following asymptotic behavior*

$$\mathbb{P}(X(t) > x) \sim \frac{\beta t}{\lambda \pi} \Gamma(\alpha) \sin\left(\frac{\pi \alpha}{2} \right) x^{-\alpha}, \text{ as } x \to \infty. \tag{23}$$

Proof. Using (21) one can obtain

$$\begin{aligned} h(x,t) &= \frac{1}{\pi x^{\alpha+1}} \sum_{j=0}^{J} (-1)^{j+1} \binom{\beta t + j}{j+1} \frac{\Gamma(j\alpha + \alpha + 1)}{\lambda^{j+1}} x^{-j\alpha} \sin\left(\frac{\pi}{2} \alpha (j+1) \right) + \mathrm{O}(x^{-N}) \\ &\sim \frac{\beta t}{\lambda \pi} \Gamma(\alpha + 1) \sin\left(\frac{\pi \alpha}{2} \right) x^{-\alpha - 1}. \end{aligned}$$

The result follows by using L'Hôpital rule. □

Remark 2.3. *One can also derive the tail behavior by using a conditioning argument and the identity* $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$ *with (3), such as (see [40])*

$$\mathbb{P}(X(t) > x) = \mathbb{E}(\mathbb{P}(S(G(t)) > x|G(t))) \sim C_\alpha x^{-\alpha}\mathbb{E}(G(t)) = \frac{(1-\alpha)}{2\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}\frac{\beta t}{\lambda}x^{-\alpha}$$
$$= \frac{\beta t}{\lambda}\frac{1}{2\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})}x^{-\alpha} = \frac{\beta t}{\lambda\pi}\Gamma(\alpha)\frac{\sin(\pi\alpha)}{2\cos(\frac{\pi\alpha}{2})}x^{-\alpha} = \frac{\beta t}{\lambda\pi}\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)x^{-\alpha}.$$

Remark 2.4. *Using the fact that* $K_\nu(\omega) \sim \sqrt{\frac{\pi}{2}}e^{-\omega}\omega^{-1/2}$ *(see [41]) as* $\omega \to \infty$, *taking under consideration (18) for* $\alpha = 2$ *we have*

$$\mathbb{P}(X(t) > x) \sim \frac{1}{2}\left(\frac{\lambda}{2}\right)^{\beta t/2} e^{-x\sqrt{2\lambda}}x^{\beta t-1}, \text{ as } x \to \infty. \tag{24}$$

Fractional moments of LLP

In this part we discuss the fractional moments for LLP. As it can be expected, the fractional moments of LLP are strictly related to fractional moments of stable and gamma distributions. The moments of stable distribution are discussed in [42].

Proposition 2.5. *For* $0 < q < \alpha < 2$, *the fractional moments of LLP have following form*

$$\mathbb{E}X^q(t) = c_{q,\alpha}\frac{\Gamma(\beta t + q/\alpha)}{\Gamma(\beta t)}\lambda^{-q/\alpha},$$

where

$$c_{q,\alpha} = \frac{2^q\Gamma(\frac{1+q}{2})\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-\frac{q}{2})\Gamma(\frac{1}{2})}. \tag{25}$$

Proof. Using self-decomposability of symmetric stable distributions, it follows

$$\mathbb{E}S^q(t) = \frac{2^q\Gamma(\frac{1+q}{2})\Gamma(1-\frac{q}{\alpha})}{\Gamma(1-\frac{q}{2})\Gamma(\frac{1}{2})}t^{q/\alpha} = c_{q,\alpha}t^{q/\alpha}.$$

Using above expression and (8) with

$$\mathbb{E}X^q(t) = \mathbb{E}S^q(G(t)) = \mathbb{E}G^{q/\alpha}(t)\mathbb{E}S^q(1),$$

the result follows. □

Remark 2.5. *Using Stirling's formula* $x^{-r}\Gamma(x+r)/\Gamma(x) \sim 1$ *as* $x \to \infty$, *one can obtain the asymptotic behavior of fractional moments for LLP, such that*

$$\mathbb{E}X^q(t) \sim c_{q,\alpha}\lambda^{-q/\alpha}(\beta t)^q, \text{ as } t \to \infty. \tag{26}$$

3. Generalization of LLP

Linnik Lévy process can be generalized if we apply the general Lévy subordinator (hereafter referred to as the subordinator) in place of a gamma subordinator in (10). We remind, the subordinator is strictly increasing Lévy process of positive values [31]. The general subordinator $D_g(t)$, $t \geq 0$ has the following LT (see [31, Section 1.3.2], [44])

$$\mathbb{E}[e^{-sD_g(t)}] = e^{-tg(s)}, s > 0, \tag{27}$$

where

$$g(s) = bs + \int_0^\infty (1 - e^{-sx})\nu(dx),\ b \geq 0, s > 0, \tag{28}$$

is the Bernstein function. The $g(s)$ function is called the Laplace exponent of the subordinator. Here b is the drift coefficient and ν is a non-negative Lévy measure on positive half-line such that

$$\int_0^\infty (x \wedge 1)\nu(dx) < \infty.$$

The assumption $\nu(0,\infty) = \infty$ guarantees that the sample paths of $D_g(t)$ are almost surely ($a.s.$) strictly increasing.
The Generalized Linnik Lévy process (GLLP) we introduce by subordinating a symmetric stable Lévy motion with the general Lévy subordinator. Thus, the GLLP $X_g(t)$ is defined as

$$X_g(t) := S(D_g(t)), \tag{29}$$

where $S(t)$ and $D_g(t)$ are independent stable Lévy motion and Lévy subordinator with LT given in (27), respectively. Using a standard conditioning argument with (2) and (27), it follows

$$\mathbb{E}(e^{iuX_g(t)}) = \mathbb{E}\left[\mathbb{E}\left(e^{iuS(D_g(t))}|D_g(t)\right)\right] = \mathbb{E}\left[e^{-|u|^\alpha D_g(t)}\right] = \exp\left(-tg(|u|^\alpha)\right). \tag{30}$$

Below we present examples of the Lévy subordinator $\{D_g(t),\ t \geq 0\}$.

Example 3.1 (Gamma subordinator)**.** *Let $\{G(t),\ t \geq 0\}$ be the gamma subordinator described in the previous sections. In this case the corresponding Laplace exponent is given by*

$$g(u) = \beta \log(1 + u/\lambda), u \geq 0. \tag{31}$$

Example 3.2 ($\tilde{\alpha}$-stable subordinator)**.** *Let $\{D_{\tilde{\alpha}}(t),\ t \geq 0\}, 0 < \tilde{\alpha} < 1$ be the $\tilde{\alpha}$-stable subordinator with LT*

$$\mathbb{E}[e^{-uD_{\tilde{\alpha}}(t)}] = e^{-tu^{\tilde{\alpha}}}, u \geq 0.$$

The corresponding Laplace exponnet is given by

$$g(u) = u^{\tilde{\alpha}}, u \geq 0. \tag{32}$$

The stable Lévy motion time-changed by an independent $\tilde{\alpha}$-stable subordinator is defined as

$$\{X^{(1)}(t)\} = \{S(D_{\tilde{\alpha}}(t)),\ t \geq 0\}.$$

Example 3.3 (Tempered $\tilde{\alpha}$-stable subordinator). *Let $\{D^{\tilde{\mu}}_{\tilde{\alpha}}(t),\ t \geq 0\}, \tilde{\mu} > 0, 0 < \tilde{\alpha} < 1$ be the tempered $\tilde{\alpha}$-stable subordinator with LT*

$$\mathbb{E}[e^{-uD^{\mu}_{\tilde{\alpha}}(t)}] = e^{-t\left((\tilde{\mu}+u)^{\tilde{\alpha}}-\tilde{\mu}^{\tilde{\alpha}}\right)}.$$

The corresponding Laplace exponent is given by

$$g(u) = (\tilde{\mu} + u)^{\tilde{\alpha}} - \tilde{\mu}^{\tilde{\alpha}}, u \geq 0. \tag{33}$$

We consider the stable Lévy motion time-changed by an independent tempered $\tilde{\alpha}$-stable subordinator, defined as

$$\{X^{(2)}(t)\} = \{S(D^{\tilde{\mu}}_{\tilde{\alpha}}(t)),\ t \geq 0\}.$$

Example 3.4 (Inverse Gaussian subordinator). *Let $\{IG(t),\ t \geq 0\}$ be the inverse Gaussian subordinator with LT (see [31, Example 1.3.21])*

$$\mathbb{E}[e^{-uIG(t)}] = e^{-t\left(\delta(\sqrt{2u+\gamma^2}-\gamma)\right)}, \quad \delta, \gamma > 0.$$

The corresponding Laplace exponent is given by

$$g(u) = \left(\delta(\sqrt{2u + \gamma^2} - \gamma)\right), u \geq 0. \tag{34}$$

Consider the stable Lévy motion time-changed by an independent inverse Gaussian subordinator, defined as

$$\{X^{(3)}(t)\} = \{S(IG(t)),\ t \geq 0\}.$$

In the next part of the paper we present main properties of the GLLP defined in (29).

Proposition 3.1. *For $\alpha \in (0, 2)$ the PDF $h_g(x, t)$ of $X_g(t)$ can be represented as*

$$h_g(x, t) = \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} ie^{-sx}e^{-tg(s^{\alpha}e^{i\pi\alpha/2})}ds,\ x > 0, \tag{35}$$

provided the function $e^{-tg(u^{\alpha})}$ is analytic in S_R given in Fig. 1.

Proof. By Fourier inversion formula we have

$$\begin{aligned} h_g(x, t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}e^{-tg(|u|^{\alpha})}du = \frac{1}{\pi}\int_0^{\infty} \cos(ux)e^{-tg(u^{\alpha})}du \\ &= \frac{1}{\pi}\mathrm{Re}\int_0^{\infty} e^{iux}e^{-tg(u^{\alpha})}du. \end{aligned} \tag{36}$$

Consider the complex region $S_R = \{u = r + is : |u| < R,\ r > 0,\ s > 0\}$ (see Fig. 1). If the function $e^{-tg(u^{\alpha})}$ is analytic in S_R. By Cauchy integral formula, we have

$$\oint_{\partial S_R} e^{iux}e^{-tg(u^{\alpha})}du = 0.$$

Let $C_R = \{u = r + is : |u| = R, r \geq 0, s \geq 0\}$ (see Figure 1), then we have

$$\int_0^R e^{irx} e^{-tg(u^\alpha)} dr + \int_{C_R} e^{iux} e^{-tg(u^\alpha)} du - \int_0^R e^{-sx} e^{-tg(s^\alpha e^{i\pi\alpha/2})} ids = 0.$$

The integral along C_R tends to 0 as $R \to \infty$ and hence

$$\int_0^R e^{irx} e^{-tg(u^\alpha)} dr = \int_0^R e^{-sx} e^{-tg(s^\alpha e^{i\pi\alpha/2})} ids.$$

Using (36) the result follows. □

Proposition 3.2. *The one dimensional PDF of GLLP satisfies the following FFPE*

$$\frac{\partial}{\partial t} h_g(x,t) = -\psi(i\partial_x) h_g(x,t),\ h_g(x,0) = \delta_0(x), \tag{37}$$

where $\psi(s) = g(|s|^\alpha)$.

Proof. Taking Fourier transform with respect to the space variable denoted by $\mathcal{F}_x$ in left hand side of the equation (37),

$$\begin{aligned}\mathcal{F}_x\left(\frac{\partial}{\partial t} h_g(x,t)\right) &= \frac{\partial}{\partial t} e^{-tg(|u|^\alpha)} \\ &= -g(|u|^\alpha)\mathcal{F}_x\left(h_g(x,t)\right).\end{aligned}$$

By inverting the Fourier transform one can obtain the desired result. □

Remark 3.1. *We can derive the tail behavior of the GLLP by using a conditioning argument and the identity* $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$ *with (3), such as (see [40])*

$$\begin{aligned}\mathbb{P}(X_g(t) > x) &= \mathbb{E}(\mathbb{P}(S(D_g(t)) > x | D_g(t))) \sim C_\alpha x^{-\alpha}\mathbb{E}(D_g(t)) \\ &= \frac{(1-\alpha)}{2\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})}\mathbb{E}(D_g(t))x^{-\alpha} \\ &= \mathbb{E}(D_g(t))\frac{1}{2\Gamma(1-\alpha)\cos(\frac{\pi\alpha}{2})}x^{-\alpha} \\ &= \frac{\mathbb{E}(D_g(t))}{\pi}\Gamma(\alpha)\frac{\sin(\pi\alpha)}{2\cos(\frac{\pi\alpha}{2})}x^{-\alpha} \\ &= \frac{\mathbb{E}(D_g(t))}{\pi}\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right)x^{-\alpha}.\end{aligned}$$

Proposition 3.3. *For* $0 < q < \alpha < 2$, *the fractional moments for GLLP have following form*

$$\mathbb{E}X^q(t) = c_{q,\alpha}\mathbb{E}[D_g^{q/\alpha}(t)],$$

where $c_{q,\alpha}$ *is gievn in (25).*

Proof. For symmetric stable process one has

$$\mathbb{E}S^q(t) = c_{q,\alpha} t^{q/\alpha},$$

with $c_{q,\alpha}$ given in (25). Using self-similarity of the stable distribution and the above expression, we obtain

$$\mathbb{E}X^q(t) = \mathbb{E}S^q(D_g(t)) = \mathbb{E}[D_g^{q/\alpha}(t)]\mathbb{E}S^q(1) = c_{q,\alpha}\mathbb{E}[D_g^{q/\alpha}(t)].$$

□

Law of iterated logarithm

Definition 3.1. *We call a function $l : (0,\infty) \to (0,\infty)$ regularly varying at 0+ with index $\nu \in \mathbb{R}$ (see [45]) if*

$$\lim_{x\to 0+} \frac{l(\lambda x)}{l(x)} = \lambda^\nu, \text{ for } \lambda > 0.$$

We first reproduce the following law of iterated logarithm (LIL) for the subordinator from [45], Chapter III, Theorem 14.

Lemma 3.1. *Let $D_g(t)$ be a subordinator with $\mathbb{E}[e^{-sD_g(t)}] = e^{-tg(s)}$, where $g(s)$ is regularly varying at $0+$ with index $\nu \in (0,1)$. Let g^{-1} be the inverse function of g and*

$$k(t) = \frac{\log\log t}{g^{-1}(t^{-1}\log\log t)}, \ (e < t).$$

Then

$$\liminf_{t\to\infty} \frac{D_g(t)}{k(t)} = \nu(1-\nu)^{(1-\nu)/\nu}, \quad a.s. \tag{38}$$

We next prove the Law of iterated logarithm for the GLLP.

Theorem 3.1 (Law of iterated logarithm)**.** *Let the Laplace exponent $g(s)$ of the subordinator $\{D_g(t)\}_{t\geq 0}$ be regularly varying at 0+ with index $\nu \in (0,1)$. Then, for $0 < \alpha < 2$ we have*

$$\liminf_{t\to\infty} \frac{X_g(t)}{(k(t))^\alpha} = \nu^{1/\alpha}\,(1-\nu)^{(1-\nu)/(\alpha\nu)}\,S(1) \quad a.s.,$$

where

$$k(t) = \frac{\log\log t}{g^{-1}(t^{-1}\log\log t)} \ (t > e),$$

and $S(t)$ is the stable Lévy motion with stability index α.

Proof. Since $S(t) \stackrel{d}{=} t^{1/\alpha}S_\alpha(1)$, we have

$$X(t) = S(D_g(t)) \stackrel{d}{=} (D_g(t))^{1/\alpha}S(1).$$

Note that $D_g(t) \to \infty$, $a.s.$ as $t \to \infty$ (see [31], Section 1.5.1). Consider now,

$$\begin{aligned}\liminf_{t\to\infty} \frac{X_g(t)}{(k(t))^{1/\alpha}} &= \liminf_{t\to\infty} \frac{S(D_g(t))}{(k(t))^{1/\alpha}} \\ &= \liminf_{t\to\infty} \frac{(D_g(t))^{1/\alpha} S(1)}{(k(t))^{1/\alpha}} \\ &= S(1) \left(\liminf_{t\to\infty} \frac{D_g(t)}{k(t)} \right)^{1/\alpha}, \ a.s. \\ &= S(1)\nu^{1/\alpha} \left(1-\nu\right)^{(1-\nu)/(\alpha\nu)} \ a.s.,\end{aligned}$$

where the last step follows from (38). □

Remark 3.2. *The results in this section hold for general subordinator. However, the law of iterated logarithm results is true for only those subordinators for which the corresponding Bernstein function is regularly varying (see Definition 3.1) with index* $\nu \in (0, 1)$*. In case of gamma subordinator, the corresponding Bernstein function (see (31))* $g(u) = \beta \log(1 + u/\lambda), u \geq 0$ *is regularly varying with index* $\nu = 1$*. Therefore, the law of iterated logarithm can not apply to LLP. Moreover, the corresponding Bernstein function of the tempered* $\tilde{\alpha}$*-stable subordinator and the inverse Gaussian subordinator are regularly varying with index* $\nu = 0$ *and* $\nu = 1$*, respectively. However, the corresponding Bernstein function of the* $\tilde{\alpha}$*-stable subordinator is regularly varying with index* $\nu = \tilde{\alpha} \in (0, 1)$*, therefore the LIL is applicable in this case.*

4. Conclusions

In this paper we have considered the LLP defined through the subordinated stable Lévy motion delayed by the gamma process. The LLP can be considered as the extension of the variance gamma process, known also as the Laplace motion. The variance gamma process was applied in various fields including physics and finance. On the other hand the introduced process is a process of independent stationary increments having Linnik distribution, one of the most physically relevant example of geometric stable distribution. In this paper we have demonstrated main properties of the LLP, like probability distribution features and law of the iterated logarithm. Finally, we have generalized the LLP to the more general Lévy processes which can be used to the description of heavy-tailed data.

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