DYNAMIC CAPACITY PLANNING AND INVENTORY CONTROL

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Abstract

Over the last three years, the COVID-19 pandemic has posed unprecedented challenges and exposed vulnerabilities in production strategies and supply chains worldwide. Apart from creating a global public health crisis, the pandemic disrupted the supply of many medically critical resources such as medical oxygen, vaccines and medicines, ventilators, personal protective equipment (PPE), and staff to care for patients. The dramatic surge in demand created severe shortages and scarcity of these resources. The problems in this thesis are motivated by the challenges in managing the medical oxygen supply chain in India. The shortage of medical oxygen in India highlighted the lack of insufficient domestic production capacity and the need for an integrated capacity-inventory strategy to respond to the crisis. In response, the government made several interventions to ease the supply constraints. The government made imports of around 50,000 metric tonnes (MT) of medical oxygen. In addition to imports and redeploying oxygen production from other industries, the government allocated investments for installing over 4,000 Pressure Swing Adsorption (PSA) plants with a daily capacity of 18,000 MT in health facilities across the country. The amount of the consumption of oxygen and the strain on the capacity of healthcare facilities treating COVID-19 patients highlighted the problem of indiscriminate and wasteful use of oxygen.

The state governments conducted oxygen audits in several hospitals and were able to reduce demand by 10-30%. But as much as rational use of oxygen is desirable, it is often challenging to control wastages, and attempting to average out requirements may become counterproductive. The demand for medical oxygen, particularly during a pandemic, is highly volatile and influenced not only by wastage in handling and delivery, but also by many randomly varying factors such as the spread of infection, change in positivity rates, enforcement of social distancing measures such as the imposition of lockdowns, the number of vaccinations administered, and so on.

In this thesis, we incorporate the relevant features such as simultaneous capacity addition, unreliable outsourcing supply, and uncertain wastage and demand from the motivating context to address the capacity and production planning problem using mathematical tools involving dynamic programming and stochastic processes.

In the first problem of this thesis, we find the optimal simultaneous capacity, production, and outsourcing strategies in the presence of an outside supplier when both supply and demand uncertainties exist. We propose a dynamic programming formulation of the problem and derive the optimal capacity addition and inventory policies structure for a finite time horizon with non-stationary demand and a random yield type uncertainty in outsourcing. We demonstrate the benefit of dynamically expanding capacity through numerical experiments under both scenarios of reliable and unreliable outsourcing supply. We find that when outsourcing is unreliable, there is a substantial benefit in dynamically adding capacity even when the lead time for capacity addition is longer. The value of the capacity addition option is higher when outsourcing is unreliable and has a non-stationary structure compared to a system where outsourcing is completely reliable.

Wastage of medical oxygen during the delivery process presents a challenging problem in the healthcare industry for managing a rising caseload during a pandemic. Wastages increase demand when the actual requirements are low, widening the demand-supply gap. In the second problem of the thesis, we focus on this crucial feature of wastage during the demand fulfillment process. We first conduct the analysis through a single-period model and use the newsvendor setting to explore the value of information about wastage on optimal decisions. Then we extend the theoretical analysis to a multiperiod setting and build a stochastic, periodic review inventory model to study the impact of wastage on capacity addition and production decisions and the total cost of running the system. We conduct numerical experiments to illustrate the structure and demonstrate the sensitivity of the optimal policy to different cost and model parameters.

Models in the inventory management literature have employed the Markovian structure to capture the impact of uncontrollable environment on demand. In the third thesis problem, we employ the specific structure of the Markov-modulated demand process to incorporate the influence of environmental and social factors. This becomes a natural extension of the second problem in the thesis. We construct a finite horizon, dynamic inventory model to find the structure of optimal production policies in a discrete-time setting. We analyse the impact of the uncertain wastage and demand state on optimal policies and cost to the decision maker.

The shortages of critical products such as medical oxygen, vaccines, PPE, and so on linked to the pandemic have highlighted the importance of building domestic production capacity to manage these resources efficiently. Industry experts are encouraging reshoring of critical supply chains. In its entirety, we expect our studies will be helpful and support governments and policymakers response with production planning and resource expansion. In general, the strategy of simultaneous optimization of capacity, production, and outsourcing decisions will be valuable in environments where dependencies on outsourcing supply are risky, shortage costs are significantly high, and the installed capacity can be used for other operations during periods of low demand, the marginal capacity cost is much lower than the effective marginal benefit it can provide over time, and the contribution margin on units produced is high. The proposed models in this thesis can also be generalized and applied to any scenario where there are high and significant chances of wastage of inventory during demand fulfillment. The insights from this study could be used in various contexts such as medical oxygen and vaccine supply chains, blood inventory management, food-bank operations and so on.

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Summary

The COVID-19 pandemic disrupted the supply and caused severe scarcity of many medically essential items such as Personal Protective Equipment (PPE), face masks, medicines, ventilators and medical oxygen, household necessities (hand sanitizer, toilet paper) and even critical automotive and electronics components (semiconductors). While the disruption due to COVID-19 pandemic was of extraordinary magnitude and scale, it is essential to note that supply chains are constantly exposed to risks stemming from natural disasters, climate change, trade-wars, geopolitical and economic uncertainties, cyber and terrorist attacks. Before the pandemic, supply chains were focused on supplying goods in a cost-effective manner relying on just-in-time systems and outsourcing and offshoring major parts of their manufacturing operations. The Covid-19 pandemic created a window of opportunity for companies to fundamentally re-evaluate their supply chains and the approach to global manufacturing and sourcing. With demand surges, unreliable supplies, reduced efficiency and increased wastages, the existing steady-state models designed to meet historically stable demand requirements proved to be inadequate. Manufacturing firms have been updating their production schedules, improving their risk-monitoring capabilities by increasing visibility to their suppliers' performance and building capacity to respond rapidly to shocks by developing new inbound routes to meet the production challenges (Butt, 2022). Industry reports and articles issued by consultancies such as BCG and Deloitte, highlight that nations worldwide are re-evaluating their outsourcing strategies to bolster resilience in supply chain of critical industries to survive future crises (Aylor et al., 2020; Rojas et al., 2022). Academic scholars of global supply chain management (Sodhi et al., 2021) have encouraged and recommended onshoring and near-shoring manufacturing operations of critical medical products.

The studies in this thesis are an attempt to respond to the calls for research for managing stockpiles, building capability and capacity of medically critical goods such as PPE, ventilators, vaccines, medicines and medical oxygen, in order to respond to massive public health emergencies such as pandemics. We take an integrated approach to determine cost-effective systems and policies that adjusts to the volatility of demand during pandemic by taking into account the tradeoff between how quickly the system can respond to demand spikes and with the cost of maintaining capacity and inventory, the uncertainty in outsourcing supply, increased wastage in the system, and the ability to dynamically expand capacity. Our models can guide decision makers to jointly optimize the three levers of capacity, production and outsourcing in a periodic review finite horizon setting. In addition to planning for medical oxygen, we list some studies from recent literature as examples where similar trade-offs may arise and these models could be applicable.

- Vaccines: In their article, Castillo et al. (2021) estimate the global benefits from vaccine capacity already in place, and find that there is substantial benefits in undertaking additional capacity investment. They urge governments and international organizations to contract with vaccine producers to further expand capacity and suggest market design to incentivize manufacturers to boost their production capacity. The progress of the pandemic and the inter-temporal trade-off between cost of delaying capacity expansion vs. higher future capacity complicates the capacity expansion decisions. Our models can be used by governments and vaccine manufacturers to determine when to expand capacity and how much. Dai and Song (2021) identify this as an important research opportunity for management scientists.

- ICU: Similar to our study, Gambaro et al. (2023) were motivated by the shortage of Covid-19-related intensive care units (ICU) capacity in 2020 in Italy. They first build a robust estimation and forecasting procedure of epidemic and demand models, and then an optimization model to support decision-makers' response in the early stages of a pandemic with ICU capacity expansion. Our studies provide a general approach for capacity expansion under uncertainty to meet the expected surge in demand for the resources.

- Critical supplies: Li et al. (2023) highlight that "Going beyond stockpiles, there is a need to identify and reserve backup domestic manufacturing capacity and develop domestic manufacturing and related capabilities". Sodhi and Tang (2021) analyse the trade-offs between inventory, capacity and capability and develop a three-tiered response system for ensuring management of supply of ventilators and PPE during rare emergencies like pandemics. Li et al. (2023) build a parsimonious two-period model of a system integrating inventory, capacity, and capability to cope with uncertain demand for consumable pharmaceuticals and medical supplies (masks and PPE), created by a pandemic or other major public health emergency.

  Our model takes a multi-period view and is an extension of their two-period model.

- Our models could also be utilised in the context of slow-onset disasters like droughts, where the planning horizon can be multiple years and each time period can correspond to months and years. Extensions of our model could address question of where to install capacity or to determine the optimal location and capacities of medical supplies for disaster preparedness in the event of a hurricanes, floods and storms.

The problem with wastage can be generalized and applied to any capacitated-production-delivery system where there is a chance of wastage during demand fulfilment such as vaccines, blood and even food delivery.
Vaccine wastage is an important component to calculate vaccine needs. According to the World Health Organisation, around 50 percent of vaccines get wasted annually during distribution (WHO, 2005). There is another aspect to wastage in vaccines that occurs due to vaccine expiry and loss of product integrity in the supply chain. Failing to accurately account for wastage may cause either vaccine shortages or excessive procurement, thereby causing more wastage through expiry. Our models can be extended and modified to factor specific features such as wastage through expiry to build efficient and effective immunization strategies and practices.
Similarly, inventory management and distribution of blood and blood products is a challenging problem for blood banks and has been extensively studied by researchers. Blood is considered as a scarce resource and the main challenge is to plan inventory in such way that shortages are minimized and wastage due to outdating and spoilage are reduced. Another natural extension of our models would be to incorporate the perishable nature of the product and develop dynamic blood ordering policies.

In particular, we have considered three problems in capacity planning and inventory control by focusing on supply side issue such as unreliable supplier, and wastage of inventory during demand fulfillment. The dynamic programming approach provides an elegant mathematical framework to analyse these problems.

Our models are a close approximation of the problem faced by the government in planning for medical oxygen in India, as described in Chapter 1. We capture several important features from the context such as the long lead time of approximately 3-4 weeks in setting up oxygen plants. Few suppliers with limited capacities and all facing unprecedented demand resulted in rationing, which is captured through a random yield uncertainty associated with supply. We also capture the phenomena of demand in pandemics or similar disruptions through a one-time demand surge in the planning horizon. This study provides optimal policies which can be used to scale oxygen generation capacities for long-term sustainability and ensure increased access to medical oxygen to serve all kind of patients requiring oxygen therapy. Specifically, our work can guide planners to answer pertinent questions such as when and how to build capacity and how much to order from the outside supplier, such that the decision maker does not carry excess production capacity and is able to meet demand during the peak periods.

As highlighted in Chapter 1, it is extremely challenging to identify wastage when there is massive strain on healthcare systems. We address the issue of planning capacity and inventory in the presence of uncontrollable wastage during demand fulfillment in hospitals, or at the point of consumption. In the second problem, our focus is on this critical aspect of wastage that arises due to clinical and operational inefficiencies. Theoretically, we provided an extension of the classic newsvendor model to account for random wastage. We demonstrate the value of incorporating and accounting for the random nature of wastage. We also analyse the wastage problem in a multi-period finite horizon setting using the dynamic programming approach.

In addition to wastage, in the third problem, we capture the impact of dynamically evolving and fluctuating environmental conditions on demand through a Markov model. Markovian structure of demand also allows the possibility of modeling patients' health conditions as discrete states, and the transition from one health state to another. These models have been used with increasing frequency in medical decision making and in epidemiological and clinical evaluations, as it provides a more accurate representation of prognosis that patients may undergo. We employ the Markov modulated demand model for economic evaluation and finding cost-effective optimal strategies.

Through our research we find that there is significant benefit in dynamically expanding capacity of critical resources such as medical oxygen. This provides evidence for policy and action to strengthen the oxygen infrastructure that can improve health outcomes and save lives.
Our studies also show that having information and accounting for wastage at the point of consumption when planning inventory leads to cost savings. However, we show that higher the wastage in the system, higher the need to add more inventory and capacity. Policy makers can consider the trade-off between building more capacity that ensures access to the resource in the long-term and exerting efforts by conducting audits to reduce wastage and demand requirements in the short-term.

It was observed that in the face of shortage of medical oxygen in India, there was a booming black market for oxygen cylinders as people struggled to get access to this life saving commodity. Medical oxygen cylinders were selling for ten times their price[1]. Rationing of medical oxygen was another challenge for the government as oxygen had to be transported from oxygen-rich states to oxygen-poor states. Many states struggled to meet demand through the quota allocated to them and urged the Centre to raise allocation[2]. In addition to regional disparities in oxygen production capacity and logistical problems, wastage of oxygen at the point of consumption inflated the

[1] `https://www.vice.com/en/article/7kv95q/india-black-market-oxygen-only-the-rich-can-survive-covid`

[2] `https://www.thehindu.com/news/cities/mumbai/raise-oxygen-allocation-by-200-mt-state-tells-centre/article34483387.ece`

requirements. The government found that some states were demanding oxygen more than their actual requirements[3]. The emergence of black markets and problems in rationing generally arise whenever there are large-scale shortages and uncertain supplies of critical products such as medical oxygen or any life saving drug.

The studies in this thesis, by providing optimal capacity and inventory policies during pandemics and disruptions in general, can have far-reaching implications by helping to circumvent these serious issues, and ensure equitable distribution of highly valuable resources.

An integrated model with capacity, production and outsourcing option, together in the presence of wastage during demand fulfilment process would provide a more realistic and representative of the motivating context. However, when considered all together, the model becomes too complicated to analyse the impact of different drivers and levers. Hence, we decided to segregate the problem into pieces to understand the individual impact of each driver. Now that we have examined the problems separately, we would be extending this work by modelling a joint, integrated scenario with all relevant aspects and mathematically solving the comprehensive model to see if the results are consistent.

While the work in this thesis takes the perspective of a central planner or a decision maker in an integrated set-up and answers questions such as when and how much, the problem can also be explored through a decentralised setting to answer questions such as where to install capacity. A decentralised view of the problem would be able to capture the spatial distribution of production plants that can serve demand in restricted geographical regions. This view opens up several avenues for future research that can addresses allocation problem, facility location problem, transportation

[3] https://www.indiatoday.in/coronavirus-outbreak/story/mumbai-delhi-oxygen-crisis-covid-cases-deaths-1799979-2021-05-07

and transshipment of inventory with time-varying demand in multi-period setting.

Further extension of our work could involve explicitly accounting for the transmission of infectious diseases in large populations and its impact on demand for medically critical items. The mathematical disease spread modelling techniques which can capture population characteristics, disease characteristics and resources constraints, will help build more effective optimal dynamic health policies such as that related to medical oxygen. These models could be solved through dynamic programming techniques but not without challenges. To solve the problem of modelling demand of medically critical items by accounting for infection spread through dynamic programming method will pose challenges two main challenges. The first challenge is the size of the state space. The size of state space in simple mathematical disease spread models such as SIR becomes prohibitively large which makes the dynamic programming methods inefficient. The other challenge is the inherent complexity in measuring the spread of disease. Asymptomatic population, challenges in diagnostic tests and other constraints create a lot of uncertainty in the actual state of the epidemic. This means that the true state of disease spread and thereby the demand of medically critical items is unknown or unobservable. Future research work can focus on solving these problems through approximate dynamic programming techniques. While we assume perfect knowledge of demand states, extensions can address the above issues through considering partially observed Markov-modulated demand models.

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Chapter 6

Appendix

6.1 Theorem 1: Convexity of $G_t(I_t, z_t, \theta_t)$

Proof of Theorem 1. We prove this by induction. From Lemma 1, we have that the one-period cost $c_M\theta_1 + c_p x_{p_1} + L(I_1 + x_{p_1})$ is convex. Since convexity is preserved under minimization operator, we have that

$$G_1(I_1, z_1, \theta_1) = \inf_{x_{p_1} \in [0, \theta_1]} \left\{ c_M\theta_1 + c_p x_{p_1} + L(I_1 + x_{p_1}) + \alpha\mathbb{E}[G_0(I_1 + x_{p_1} - \xi_1, z_0, \theta_1 + v_0)] \right\} \textit{ is convex.}$$

$$G_2(I_2, z_2, \theta_2) = \inf_{x_{p_2} \in [0, \theta_2]} \Big\{ c_M\theta_2 + c_p x_{p_2} + L(I_2 + x_{p_2}) + \inf_{x_{s_2} \geq 0} \Big\{ c_s x_{s_2} + \alpha\mathbb{E}[G_1(I_2 + x_{p_2} + x_{s_2} - \xi_2, z_1, \theta_2 + v_1)] \Big\} \Big\}$$

Since $G_1(.,.,.)$ is convex, so is $\alpha\mathbb{E}G_1(.,.,.)$ because convexity is preserved under linear transformation and expectation.

$\implies c_s x_{s_2} + \alpha\mathbb{E}[G_1(.,.,.)]$ is convex.

$\implies \inf_{x_{s_2}\geq 0}\left\{c_s x_{s_2} + \alpha\mathbb{E}[G_1(.,.,.)]\right\}$ is convex, as convexity is preserved under minimization.

Then, $\inf_{x_{p_2}\in[0,\theta_2]}\left\{c_M\theta_2 + c_p x_{p_2} + L(I_2+x_{p_2}) + \inf_{x_{s_2}\geq 0}\left\{c_s x_{s_2} + \alpha\mathbb{E}[G_1(.,.,.)]\right\}\right\}$ is convex because of the fact that sum of two convex functions is convex and infimum preserves convexity. Then, for $t \geq l$,

$$G_t(I_t, z_t, \theta_t) = \inf_{v_{t-l}\geq 0}\left\{c_c v_{t-l} + c_M\theta_t + \inf_{x_{p_t}\in[0,\theta_t]}\left\{c_p x_{p_t} + L(I_t + x_{p_t}) + \inf_{x_{s_t}\geq 0}\left\{c_s x_{s_t} + \alpha\mathbb{E}[G_{t-1}(.,.,.)]\right\}\right\}\right\}$$

is convex, using the same properties of convexity. It directly follows that $G_t(I_t, z_t, \theta_t)$ is convex for all t. □

6.2 Theorem 2: Convexity of $G_t^u(I_t, z_t, \theta_t)$

Proof of Theorem 2. We prove this by induction. From Lemma 1, we have that the last-period cost function $c_M\theta_1 + c_p x_{p_1} + L(I_1 + x_{p_1})$ is convex. Since convexity is preserved under minimization, $G_1^u(I_1, z_1, \theta_1) = \inf_{0\leq x_{p_1}\leq\theta_1}\{c_M\theta_1 + c_p(x_{p_1} - I_1) + L(I_1 + x_{p_1})\}$ is convex.

Continuing with induction, we show that $G_2^u(I_2, z_2, \theta_2)$ is convex.

$$G_2^u(I_2, z_2, \theta_2) = \inf_{x_{p_2}\in[0,\theta_2], x_{s_2}\geq 0}\Big\{c_M\theta_2 + c_p x_{p_2} + L(I_2 + x_{p_2})$$
$$+\alpha c_s\mu_2 x_{s_2} + \alpha\mathbb{E}_{\xi_2}\mathbb{E}_{\beta_2}[G_1^u(I_2 + x_{p_2} + \beta_2 x_{s_2} - \xi_2, z_1, \theta_2 + v_1)]\Big\}$$

Since $I_2+x_{p_2}+\beta_2 x_{s_2}-\xi_2$ and θ_2+v_1 are linear in all the terms, then by Theorem 5.7 of Rockafellar (2015) and the fact that $G_1^u(.,.,.)$ is convex, we get that $G_1^u(I_2+x_{p_2}+\beta_2 x_{s_2}-\xi_2, z_1, \theta_2+v_1)$ is convex. Since convexity is preserved under expectation, we have that $\alpha\mathbb{E}_{\xi_2}\mathbb{E}_{\beta_2}[G_1^u(I_2+x_{p_2}+\beta_2 x_{s_2}-\xi_2, z_1, \theta_2+v_1)]$ is convex. As sum of convex functions is convex, then $c_M\theta_2+c_p x_{p_2}+L(I_2+x_{p_2})+\alpha c_s\mu_2 x_{s_2}+\alpha\mathbb{E}_{\xi_2}\mathbb{E}_{\beta_2}[G_1^u(I_2+x_{p_2}+\beta_2 x_{s_2}-\xi_2, z_1, \theta_2+v_1)]$ is convex. Since convexity is preserved under infimum operator, we have that $\inf_{x_{p_2}\in[0,\theta_2], x_{s_2}\geq 0}\Big\{c_M\theta_2+c_p x_{p_2}+L(I_2+x_{p_2})+\alpha c_s\mu_2 x_{s_2}+\alpha\mathbb{E}_{\xi_2}\mathbb{E}_{\beta_2}[G_1^u(I_2+x_{p_2}+\beta_2 x_{s_2}-\xi_2, z_1, \theta_2+v_1)]\Big\}$ is convex. Therefore, $G_2^u(I_2, z_2, \theta_2)$ is convex. Then, by the same properties of convex functions, it can be easily shown that for $t\geq l$, $G_t^u(I_t, z_t, \theta_t)$ is convex. Now, we assume that $G_{t-1}^u(I_{t-1}, z_{t-1}, \theta_{t-1})$ is convex for any t. Using the induction hypothesis and the arguments used in the base case, it directly follows that $G_t^u(I_t, z_t, \theta_t)$ is convex. □

6.3 Theorem 3: Structure of optimal policy under reliable outsourcing

Proof of Theorem 3. Substitute $z=\theta_t+v_{t-1}$, $x=I_t+x_{p_t}$, and $y=I_t+x_{p_t}+x_{s_t}$, rewriting equation (2.4)

$$\Gamma_t(I_t, z_t, z) = \inf_{x\in[I_t, I_t+\theta_t], y\geq x}\Big\{c_p(x-I_t)+L(x)+c_s(y-x)+\alpha\mathbb{E}[G_{t-1}(y-\xi_t, z_{t-1}, z)]\Big\} \tag{6.1}$$

$$= \inf_{x\in[I_t, I_t+\theta_t]}\Big\{(c_p-c_s)(x-I_t)+L(x)+\inf_{y\geq x}\Big\{c_s(y-I_t)+\alpha\mathbb{E}[G_{t-1}(y-\xi_t, z_{t-1}, z)]\Big\}\Big\}$$

$$= \inf_{x\in[I_t, I_t+\theta_t]}\Big\{(c_p-c_s)(x-I_t)+L(x)+\tilde{L}(x)\Big\} \tag{6.2}$$

where

$$\tilde{L}(x) = \inf_{y \geq x} \{c_s(y - I_t) + \alpha \mathbb{E}[G_{t-1}(y - \xi_t, z_{t-1}, z)]\} \tag{6.3}$$

Note that $\tilde{L}(x)$ is a convex function in y. Let S_t be the minimizer of $\tilde{L}(x)$.

Also, $(c_p - c_s)(x - I_t) + L(x) + \tilde{L}(x)$ is a convex function in x, there exists a minimizer B_t such that

$$\Gamma_t(I_t, z_t, z) = \begin{cases} (c_p - c_s)\theta_t + L(I_t + \theta_t) + \tilde{L}(I_t + \theta_t), & I_t < B_t - \theta_t \\ (c_p - c_s)(B_t - I_t) + L(B_t) + \tilde{L}(B_t), & B_t - \theta_t \leq I_t < B_t \\ L(I_t) + \tilde{L}(I_t), & I_t \geq B_t \end{cases} \tag{6.4}$$

Case 1: $I_t \geq B_t$

$$\begin{aligned} \Gamma_t(I_t, z_t, z) &= L(I_t) + \tilde{L}(I_t) \qquad (6.5) \\ &= L(I_t) + \inf_{y \geq I_t} \{c_s(y - I_t) + \alpha \mathbb{E}[G_{t-1}(y - \xi_t, z_{t-1}, z)]\} \\ &= L(I_t) + \begin{cases} c_s(S_t - I_t) + \alpha \mathbb{E}[G_{t-1}(S_t - \xi_t, z_{t-1}, z)], & I_t < S_t \\ \alpha \mathbb{E}[G_{t-1}(I_t - \xi_t, z_{t-1}, z)], & I_t \geq S_t \end{cases} \end{aligned}$$

Case 2: $B_t - \theta_t \leq I_t \leq B_t$

$$\begin{aligned}
\Gamma_t(I_t, z_t, z) &= (c_p - c_s)(B_t - I_t) + L(B_t) + \tilde{L}(B_t) \qquad (6.6)\\
&= (c_p - c_s)(B_t - I_t) + L(B_t) + \inf_{y \geq B_t \geq I_t} \{c_s(y - I_t) + \alpha\mathbb{E}[G_{t-1}(y - \xi_t, z_{t-1}, z)]\}\\
&= c_p(B_t - I_t) + L(B_t) + \inf_{y \geq B_t} \{c_s(y - B_t) + \alpha\mathbb{E}[G_{t-1}(y - \xi_t, z_{t-1}, z)]\}\\
&= c_p(B_t - I_t) + L(B_t) + \begin{cases} c_s(S_t - B_t) + \alpha\mathbb{E}[G_{t-1}(S_t - \xi_t, z_{t-1}, z)], & B_t < S_t \\ \alpha\mathbb{E}[G_{t-1}(B_t - \xi_t, z_{t-1}, z)], & B_t \geq S_t \end{cases}
\end{aligned}$$

Case 3: $I_t \leq B_t - \theta_t$

$$\begin{aligned}
\Gamma_t(I_t, z_t, z) &= (c_p - c_s)\theta_t + L(I_t + \theta_t) + \tilde{L}(I_t + \theta_t) \qquad (6.7)\\
&= (c_p - c_s)\theta_t + L(I_t + \theta_t) + \inf_{y \geq I_t + \theta_t} \{c_s(y - I_t) + \alpha\mathbb{E}[G_{t-1}(y - \xi_t, z_{t-1}, z)]\}\\
&= c_p\theta_t + L(I_t + \theta_t) + \inf_{y \geq I_t + \theta_t} \{c_s(y - I_t - \theta_t) + \alpha\mathbb{E}[G_{t-1}(y - \xi_t, z_{t-1}, z)]\}\\
&= c_p\theta_t + L(I_t + \theta_t) + \begin{cases} c_s(S_t - I_t - \theta_t) + \alpha\mathbb{E}[G_{t-1}(S_t - \xi_t, z_{t-1}, z)], & I_t + \theta_t < S_t \\ \alpha\mathbb{E}[G_{t-1}(I_t + \theta_t - \xi_t, z_{t-1}, z)], & I_t + \theta_t \geq S_t \end{cases}
\end{aligned}$$

Summarizing the three cases above, the optimal order quantities are:

$$x^*_{p_t} = max\Big\{0, min\Big\{\theta_t, B_t - I_t\Big\}\Big\}$$
$$x^*_{s_t} = max\Big\{0, S_t - (I_t + x^*_{p_t})\Big\}$$

To characterize the capacity addition policy for $t \geq l$, we can rewrite (2.5) as

$$G_t(I_t, z_t, \theta_t) = inf_{v_{t-l} \geq 0}\{c_c v_{t-l} + c_M \theta_t + \Gamma_t(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l})\}$$

Define $h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l}) = c_c v_{t-l} + c_M \theta_t + \Gamma_t(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l})$. From Theorem 1 we have that h is convex.

$$\frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l})}{\partial v_{t-l}} = c_c + \Gamma'_t(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l})$$

Let $L_t = inf\{z_t : \frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0)}{\partial v_{t-l}} \geq 0\}$.

Then for $z_t \geq L_t$

$$\begin{aligned}
\frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l})}{\partial v_{t-l}} &\geq \frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0)}{\partial v_{t-l}} \\
&= c_c + \Gamma'_t(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0) \\
&\geq c_c + \Gamma'_t(x^*_{p_t}, x^*_{s_t}, I_t, L_t, \theta_t, 0) \\
&= \frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, L_t, \theta_t, 0)}{\partial v_{t-l}} \\
&= 0
\end{aligned}$$

$\implies v^*_{t-l} = 0$.

Since for $z_t < L_t$, $\frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0)}{\partial v_{t-l}} < 0 \implies v^*_{t-l} > 0$. Therefore, the optimal capacity addition policy is

$$v_{t-l}^{*} = \begin{cases} > 0, & z_t < L_t \\ 0, & z_t \geq L_t \end{cases}$$

□

6.4 Theorem 4: Structure of optimal policy under unreliable outsourcing

Proof of Theorem 4.

$$\frac{\partial \Gamma_t^u(x_{p_t}, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} = c_p + L^{'}(I_t + x_{p_t}) + \alpha \mathbb{E}_{\xi_t} \mathbb{E}_{\beta_t}[G^{u'}_{t-1}(I_t + x_{p_t} + \beta_t x_{s_t} - \xi_t, z_{t-1}, \theta_t + v_{t-1})]$$
$$\frac{\partial \Gamma_t^u(x_{p_t}, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} = \mathbb{E}_{\beta_t}[\alpha \beta_t c_s + \alpha \beta_t \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(I_t + x_{p_t} + \beta_t x_{s_t} - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$
$$\frac{\partial \Gamma_t^u(0, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} = c_p + L^{'}(I_t) + \alpha \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(I_t - \xi_t, z_{t-1}, \theta_t + v_{t-1})]$$
$$\frac{\partial \Gamma_t^u(0, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} = \mu_t[\alpha c_s + \alpha \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(I_t - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$

Let s_1 and s_2 respectively satisfy

$$s_1 = inf\{I_t : \frac{\partial \Gamma_t^u(0, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} \geq 0\}$$
$$s_2 = inf\{I_t : \frac{\partial \Gamma_t^u(0, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} \geq 0\}$$

If $s_1 < s_2$, then let $S_t^u = s_2$.

For $I_t > S_t^u$,

$$\frac{\partial \Gamma_t^u(x_{p_t}, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} \geq \frac{\partial \Gamma_t^u(0, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}}$$
$$= \mu_t[\alpha c_s + \alpha \mathbb{E}_{\xi_t}[{G^{u}}'_{t-1}(I_t - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$
$$\geq \mu_t[\alpha c_s + \alpha \mathbb{E}_{\xi_t}[{G^{u}}'_{t-1}(S_t^u - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$
$$= 0$$

$\implies x_{s_t}^* = 0$ for $I_t > S_t^u$. Similarly, $x_{p_t}^* = 0$ for $I_t > S_t^u$.

For $I_t < S_t^u$,

$x_{s_t}^* > 0$ for $I_t < S_t^u$, since

$$\frac{\partial \Gamma_t^u(0, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} = \mathbb{E}_{\beta_t}[\alpha \beta_t c_s + \alpha \beta_t \mathbb{E}_{\xi_t}[{G^{u}}'_{t-1}(I_t + \beta_t x_{s_t} - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$
$$= 0$$
$$\implies -\alpha c_s \mu_t = \mathbb{E}_{\beta_t}[\alpha \beta_t \mathbb{E}_{\xi_t}[{G^{u}}'_{t-1}(I_t + \beta_t x_{s_t} - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$
$$\geq \mu_t \mathbb{E}_{\beta_t}[\alpha \mathbb{E}_{\xi_t}[{G^{u}}'_{t-1}(I_t + \beta_t x_{s_t} - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$

The last inequality is due to the property $\int \phi(x)\psi(x)f(x)dx \geq (\int \phi(x)f(x)dx)(\int \psi(x)f(x)dx)$.

$$\implies \mathbb{E}_{\beta_t}[\alpha \beta_t c_s] + \alpha \mathbb{E}_{\beta_t}[\beta_t] \mathbb{E}_{\xi_t}[{G^{u}}'_{t-1}(I_t + \beta_t x_{s_t} - \xi_t, z_{t-1}, \theta_t + v_{t-1})] < 0$$
$$\implies \frac{\partial \Gamma_t^u(0, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} < 0$$
$$\implies x_{s_t}^* > 0$$

Let B_t^u satisfy $\frac{\partial \Gamma_t^u(0, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} = 0$.

Then for $B_t^u < I_t < S_t^u$,

$$\begin{aligned}
\frac{\partial \Gamma_t^u(x_{p_t}, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} &\geq \frac{\partial \Gamma_t^u(0, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} \\
&= c_p + L^{'}(I_t) + \alpha \mathbb{E}_{\xi_t}[{G^{u}}^{'}_{t-1}(I_t + \beta_t x_{s_t}^* - \xi_t, z_{t-1}, \theta_t + v_{t-1})] \\
&\geq c_p + L^{'}(B_t^u) + \alpha \mathbb{E}_{\xi_t}[{G^{u}}^{'}_{t-1}(B_t^u + \beta_t x_{s_t}^* - \xi_t, z_{t-1}, \theta_t + v_{t-1})] \\
&\geq 0
\end{aligned}$$

$\implies x_{p_t}^* = 0$.

By definition of B_t^u, we have that for $I_t < B_t^u$, $x_{p_t}^*, x_{s_t}^* > 0$. So the optimal production and outsourcing policies are characterized as:

$$(x_{p_t}^*, x_{s_t}^*) = \begin{cases} (\theta_t - I_t, x_{s_t}^*), & I_t < B_t^u - \theta_t \\ (x_{p_t}^*, x_{s_t}^*), & B_t^u - \theta_t \leq I_t < B_t^u \\ (0, x_{s_t}^*), & B_t^u \leq I_t < S_t^u \\ (0, 0), & I_t \geq S_t^u \end{cases}$$

If $s_1 > s_2$. Let $B_t^u = s_1$.

For $I_t > B_t^u$

$$\frac{\partial \Gamma_t^u(x_{p_t}, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} \geq \frac{\partial \Gamma_t^u(0, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}}$$

$$= c_p + L^{'}(I_t) + \alpha \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(I_t - \xi_t, z_{t-1}, \theta_t + v_{t-1})]$$

$$\geq c_p + L^{'}(B_t^u) + \alpha \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(B_t^u - \xi_t, z_{t-1}, \theta_t + v_{t-1})]$$

$$= 0$$

$\implies x_{p_t}^* = 0$ for $I_t > B_t^u$. Similarly, $x_{s_t}^* = 0$ for $I_t > B_t^u$.

For $I_t < B_t^u$,

$x_{p_t}^* > 0$ for $I_t < B_t^u$, since

$$\frac{\partial \Gamma_t^u(x_{p_t}, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} < 0$$

Let S_t^u satisfy $\frac{\partial \Gamma_t^u(x_{p_t}^*, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} = 0$ where $x_{p_t}^*$ satisfies $\frac{\partial \Gamma_t^u(x_{p_t}^*, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{p_t}} = 0$

Then if $S_t^u \leq I_t < B_t^u$,

$$\frac{\partial \Gamma_t^u(x_{p_t}^*, x_{s_t}, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}} \geq \frac{\partial \Gamma_t^u(x_{p_t}^*, 0, I_t, z_t, \theta_t, v_{t-1})}{\partial x_{s_t}}$$

$$= \mathbb{E}_{\beta_t}[\alpha \beta_t c_s + \alpha \beta_t \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(I_t + x_{p_t}^* - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$

$$\geq \mathbb{E}_{\beta_t}[\alpha \beta_t c_s + \alpha \beta_t \mathbb{E}_{\xi_t}[G^{u'}_{t-1}(S_t^u + x_{p_t}^* - \xi_t, z_{t-1}, \theta_t + v_{t-1})]]$$

$$= 0$$

$\implies x_{s_t}^* = 0.$

By definition of S_t^u, we have that for $I_t < S_t^u$, $x_{p_t}^*, x_{s_t}^* > 0$. Based on relative position of $B_t^u - \theta_t$ the optimal production and outsourcing policies are characterized as:

If $S_t^u < B_t^u - \theta_t < B_t^u$, then

$$(x_{p_t}^*, x_{s_t}^*) = \begin{cases} (\theta_t - I_t, x_{s_t}^*), & I_t < S_t^u \\ (\theta_t - I_t, 0), & S_t^u \le I_t < B_t^u - \theta_t \\ (x_{p_t}^*, 0), & B_t^u - \theta_t \le I_t < B_t^u \\ (0, 0), & I_t \ge B_t^u \end{cases}$$

If $B_t^u - \theta_t < S_t^u < B_t^u$, then

$$(x_{p_t}^*, x_{s_t}^*) = \begin{cases} (\theta_t - I_t, x_{s_t}^*), & I_t < B_t^u - \theta_t \\ (x_{p_t}^*, x_{s_t}^*), & B_t^u - \theta_t \le I_t < S_t^u \\ (x_{p_t}^*, 0), & S_t^u \le I_t < B_t^u \\ (0, 0), & I_t \ge B_t^u \end{cases}$$

We now derive the optimal capacity addition policy.

$$G_t^u(I_t, z_t, \theta_t) = inf_{v_{t-l} \ge 0}\{c_c v_{t-l} + c_M \theta_t + \Gamma_t^u(x_{p_t}^*, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-l})\}$$

Define $h(x_{p_t}^*, x_{s_t}^*, I_t, \theta_t, v_{t-l}) = c_c v_{t-l} + c_M \theta_t + \Gamma_t^u(x_{p_t}^*, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-l})$. From Theorem 2 we have that h is convex.

$$\frac{\partial h(x_{p_t}^*, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-l})}{\partial v_{t-l}} = c_c + {\Gamma^u}'_t(x_{p_t}^*, x_{s_t}^*, I_t, z_t, \theta_t, v_{t-l})$$

Let $L_t^u = inf\{z_t : \frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0)}{\partial v_{t-l}} \geq 0\}$.

Then for $z_t \geq L_t^u$

$$\begin{aligned}
\frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, v_{t-l})}{\partial v_{t-l}} &\geq \frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0)}{\partial v_{t-l}} \\
&= c_c + {\Gamma^{u}}'_t(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0) \\
&\geq c_c + {\Gamma^{u}}'_t(x^*_{p_t}, x^*_{s_t}, I_t, L_t^u, \theta_t, 0) \\
&= \frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, L_t^u, \theta_t, 0)}{\partial v_{t-l}} \\
&= 0
\end{aligned}$$

$\implies v^*_{t-l} = 0$.

Since for $z_t < L_t^u$, $\frac{\partial h(x^*_{p_t}, x^*_{s_t}, I_t, z_t, \theta_t, 0)}{\partial v_{t-l}} < 0 \implies v^*_{t-l} > 0$. Therefore, the optimal capacity addition policy is

$$v^*_{t-l} = \begin{cases} > 0, & z_t < L_t^u \\ 0, & z_t \geq L_t^u \end{cases}$$

□

6.5 Lemma 2: For supermodularity

Proof of Lemma 2. Denote the global optimum of g as $\hat{y}$.

To show: For all $x_1 \leq x_2$ and $\theta_1 \leq \theta_2$

$$H(x_2, \theta_1) + H(x_1, \theta_2) \leq H(x_1, \theta_1) + H(x_2, \theta_2) \tag{6.8}$$

$$H(x, \theta) = \begin{cases} g(x) \ \textit{increasing in } x, & \hat{y} < x \\ g(\hat{y}) \ \textit{constant}, & \hat{y} \leq x + \theta \ \textit{and} \ x < \hat{y} \\ g(x+\theta) \ \textit{decreasing in } x + \theta, & x + \theta \leq \hat{y} \end{cases}$$

Claim: $H(x_1, \theta_2) \leq H(x_1, \theta_1)$ for all x_1, θ_1 and θ_2 when $\theta_1 \leq \theta_2$ because

Case (a): $\hat{y} \leq x_1 \implies H(x_1, \theta_2) = H(x_1, \theta_1)$

Case (b): $\hat{y} \leq x_1 + \theta_1 \ \textit{and} \ x_1 \leq \hat{y} \implies H(x_1, \theta_2) = H(x_1, \theta_1)$

Case (c): $x_1 \leq \hat{y}, \ \ x_1 + \theta_1 \leq \hat{y} \ \textit{and} \ x_1 + \theta_2 \geq \hat{y} \implies H(x_1, \theta_2) \leq H(x_1, \theta_1)$

Case (d): $x_1 + \theta_2 \leq \hat{y} \implies H(x_1, \theta_2) \leq H(x_1, \theta_1)$

When $x_2 \geq \hat{y}$, $H(x_2, \theta_1) = H(x_2, \theta_2)$ This implies that 6.8 holds.

When $x_2 \leq \hat{y}$, define a single-variable function

$$h(x + \theta) = H(x, \theta)$$

Note that h is convex. From Lemma 2.6.2 in Topkis (2011) we have that if $a_1, a_2 \in \mathbb{R}^+$, and $g(y)$ is convex, then $f(x) = g(a_1 x_1 + a_2 x_2)$ is supermodular in $x \in \mathbb{R}^{\not\geq}$. $\implies$ H is supermodular as it is defined as a single argument convex function h with its arguments as non-negative linear combination. □

6.6 Lemma 3: For supermodularity

Proof of Lemma 3. Let $x_1 \leq x_2$ and $z_1 \leq z_2$

Define $y_A = arg\min_{y \geq x_1} g(y, z_1)$ and $y_B = arg\min_{y \geq x_2} g(y, z_2)$.

Then we can write $H(x_1, z_1) = g(y_A, z_1)$ and $H(x_2, z_2) = g(y_B, z_2)$

Now since

$$y_A \geq x_1 \implies \inf_{y \geq x_1} g(y, z_2) = H(x_1, z_2) \leq g(y_A, z_2)$$

$$and\ y_B \geq x_2 \implies \inf_{y \geq x_2} g(y, z_1) = H(x_2, z_1) \leq g(y_B, z_1)$$

If $y_A \leq y_B$, then by the supermodularity of g, we have that,

$$\begin{aligned} H(x_1, z_2) + H(x_2, z_1) &\leq g(y_A, z_2) + g(y_B, z_1) \\ &\leq g(y_A, z_1) + g(y_B, z_2) \\ &= H(x_1, z_1) + H(x_2, z_2) \end{aligned}$$

Hence, H is supermodular.

If $y_A \geq y_B$, then $y_A, y_B \geq x_2$

$$\implies \inf_{y \geq x_2} g(y, z_1) = H(x_2, z_1) = g(y_A, z_1)$$

$$and\ \inf_{y \geq x_1} g(y, z_2) = H(x_1, z_2) \leq g(y_B, z_2)$$

Then we have that,

$$H(x_2, z_1) + H(x_1, z_2) \leq g(y_A, z_1) + g(y_B, z_2)$$
$$= H(x_1, z_1) + H(x_2, z_2)$$

Hence, H is supermodular.

□

6.7 Proposition 3: Optimal production quantity under partial information on the probability distribution of wastage

Proof of Proposition 3. From equation (3.19) we can write,

$$\frac{d[cy + L(y)]}{dy} = c + h \int_0^1 \frac{y}{d(1 + (\lambda_H - \lambda_L)\theta + \lambda_L)} d\theta - p \int_0^1 \frac{1}{(1 + (\lambda_H - \lambda_L)\theta + \lambda_L)} \left(1 - \frac{y}{d(1 + (\lambda_H - \lambda_L)\theta + \lambda_L)}\right) d\theta = 0 \tag{6.9}$$

Solving this we get,

$$y^* = \left[\frac{\frac{p}{(\lambda_H - \lambda_L)} \, ln\left(\frac{1+\lambda_H}{1+\lambda_L}\right) - c}{\frac{p}{(1+\lambda_H)(1+\lambda_L)} + \frac{h}{(\lambda_H - \lambda_L)} \, ln\left(\frac{1+\lambda_H}{1+\lambda_L}\right)} \right] d \tag{6.10}$$

□

6.8 Theorem 6: Convexity and supermodularity of $G_t(x, \theta, \lambda, Z)$

Proof of Theorem 6. By Induction: Given capacity θ, we first find the optimal production policy

$$a_t(y, \theta, \lambda, Z) = \phi(y, \lambda) + \alpha \mathbb{E} G_{t-1}\left(f\left(y, \xi_t, \lambda\right), \theta, \lambda, Z\right)$$

Since G_{t-1} is convex (by Induction Hypothesis) and $\phi(y, \lambda)$ is linear, so a_t is convex.

Let $S_t(\theta, \lambda)$ denote the unique minimizer over $y \in [0, \infty)$. Therefore, the optimal production policy is to get as close to $S_t(\theta, \lambda)$ as possible within $[x, x + \theta]$.

Since G_{t-1} is supermodular in (x, θ) (by Induction Hypothesis), so is a_t and therefore $S_t(\theta, \lambda)$ is decreasing in θ.

Since the cost function is linear (convex function) and $\{(x, y, \theta) : x \geq 0, 0 \leq \theta \leq Z, x \leq y \leq x + \theta\}$ is a convex set, hence it is straightforward from the convexity preservation theorem under minimization that $\Gamma_t(x, \theta, \lambda, Z)$ and $G_t(x, \theta, \lambda, Z)$ are (jointly) convex in (x, θ).

To show: Γ_t is supermodular

$$\Gamma_t(x, \theta, \lambda, Z) = \begin{cases} a_t(x, \theta, \lambda, Z) & S_t(\theta, \lambda) < x \\ a_t(S_t(\theta, \lambda), \theta, \lambda, Z) & x \leq S_t(\theta, \lambda) \leq x + \theta \\ a_t(x + \theta, \theta, \lambda, Z) & x + \theta \leq S_t(\theta, \lambda) \end{cases} \tag{6.11}$$

Partial derivative wrt x,

$$\Gamma_t^{(1)}(x,\theta,\lambda,Z) = \begin{cases} a_t^{(1)}(x+\theta,\theta,\lambda,Z) & x \le S_t(\theta,\lambda) and \theta \le S_t(\theta,\lambda) - x \\ 0 & x \le S_t(\theta,\lambda) and \theta > S_t(\theta,\lambda) - x \\ a_t^{(1)}(x,\theta,\lambda,Z) & x > S_t(\theta,\lambda) \end{cases} \tag{6.12}$$

To show: $\Gamma_t^{(1)}(x,\theta,\lambda,Z)$ is increasing in θ. We know that, $a_t(S_t(\theta,\lambda),\theta,\lambda,Z) = 0$.

When x and $x+\theta < S_t(\theta,\lambda) \implies a_t^{(1)}(x+\theta,\theta,\lambda,Z) \le 0$. When $x > S_t(\theta,\lambda) \implies a_t^{(1)}(x,\theta,\lambda,Z) \ge 0$. Hence, $\Gamma_t^{(1)}(x,\theta,\lambda,Z)$ is increasing in $\theta \implies \Gamma_t$ is supermodular.

Since, G_t is convex and $g_t(x,\theta,\lambda,Z) = c_c\theta + \Gamma_t(x,\theta,\lambda,Z)$. Let $L_t(x,Z)$ be the minimizer of g_t over $\theta \in [0,Z]$. Since g_t is supermodular, $L_t(x,Z)$ is decreasing in x.

$$G_t(x,\theta,\lambda,Z) = \begin{cases} -c_p x - c_c\theta + c_c L_t(x,Z) + \Gamma_t(x, L_t(x,Z),\lambda,Z) & \theta \le L_t(x,Z) \\ -c_p x + \Gamma_t(x,\theta,\lambda,Z) & \theta > L_t(x,Z) \end{cases} \tag{6.13}$$

$$= \begin{cases} -c_p x + c_c(L_t(x,Z) - \theta) + \Gamma_t(x, L_t(x,Z),\lambda,Z) & \theta \le L_t(x,Z) \\ -c_p x + \Gamma_t(x,\theta,\lambda,Z) & \theta > L_t(x,Z) \end{cases} \tag{6.14}$$

$$G_t^{(1)}(x,\theta,\lambda,Z) = \begin{cases} -c_c & \theta \le L_t(x,Z) \\ \Gamma_t^{(1)}(x,\theta,\lambda,Z) & \theta > L_t(x,Z) \end{cases} \tag{6.15}$$

To show: G_t is supermodular, that is, $G_t^{(1)}$ is increasing in x for each fixed θ.

When $\theta > L_t(x,Z)$, $G_t^{(1)}(x,\theta,\lambda,Z)$ increases in x. Since $\Gamma_t(x,\theta,\lambda,Z)$ is supermodular, so $\Gamma_t^{(1)}(x,\theta,\lambda,Z)$ increases in x.

Hence, G_t is supermodular in (x,θ). □

6.9 Theorem 7: $G_t(x, \theta|\lambda)$ increases in λ

Proof of Theorem 7. Let $S_1^*(v|\lambda)$ and $L_1^*(x|\lambda)$ be optimal levels of production and capacity for given λ. We prove that $G_t(x, \theta|\lambda)$ increases in λ through induction on t. We first show that $G_1(x, \theta|\lambda)$ increases in λ. Pick a pair of values such that $\lambda_2 > \lambda_1$.

$$
\begin{aligned}
G_1(x, \theta|\lambda_1) &= \inf_{Z \geq v \geq \theta} \{-c_p x - c_c \theta + g_1(x, v, Z|\lambda_1)\} \\
&\leq -c_p x - c_c \theta + g_1(S_1^*(v|\lambda_2), L_1^*(x|\lambda_2), Z|\lambda_1) \\
&= -c_p x - c_c \theta + c_c L_1^*(x|\lambda_2) + c_p S_1^*(v|\lambda_2) + L(S_1^*(v|\lambda_2), \lambda_2|\lambda_1) \\
&\leq -c_p x - c_c \theta + c_c L_1^*(x|\lambda_2) + c_p S_1^*(v|\lambda_2) + L(S_1^*(v|\lambda_2), \lambda_2|\lambda_2) \\
&= G_1(x, \theta|\lambda_2)
\end{aligned}
\tag{6.16}
$$

The first inequality is due to the fact that $S_1^*(v|\lambda_2)$ and $L_1^*(x|\lambda_2)$ are suboptimal pair of inventory and capacity when $\lambda = \lambda_1$. The second inequality is due to the fact that the holding and shortage cost function $L(x, \lambda)$ is increasing in λ.

Suppose $G_k(x,\theta|\lambda)$ is increasing in λ for $k = 1, 2, \ldots, t-1$. Then

$$
\begin{aligned}
G_t(x,\theta|\lambda_1) &= \inf_{Z\geq v\geq\theta}\{-c_p x - c_c\theta + g_t(x, v, Z|\lambda_1)\} \\
&\leq -c_p x - c_c\theta + g_t(S_t^*(v|\lambda_2), L_t^*(x|\lambda_2), Z|\lambda_1) \\
&= c_c(L_t^*(x|\lambda_2) - \theta) + c_p(S_t^*(v|\lambda_2) - x) + L(S_t^*(v|\lambda_2), \lambda_2|\lambda_1) \\
&+ \alpha\mathbb{E}G_{t-1}(f(S_t^*(v|\lambda_2), \xi_t, \lambda_1), L_t^*(x|\lambda_2), Z|\lambda_1) \\
&\leq c_c(L_t^*(x|\lambda_2) - \theta) + c_p(S_t^*(v|\lambda_2) - x) + L(S_t^*(v|\lambda_2), \lambda_2|\lambda_1) \\
&+ \alpha\mathbb{E}G_{t-1}(f(S_t^*(v|\lambda_2), \xi_t, \lambda_1), L_t^*(x|\lambda_2), Z|\lambda_2) \\
&\leq c_c(L_t^*(x|\lambda_2) - \theta) + c_p(S_t^*(v|\lambda_2) - x) + L(S_t^*(v|\lambda_2), \lambda_2|\lambda_2) \\
&+ \alpha\mathbb{E}G_{t-1}(f(S_t^*(v|\lambda_2), \xi_t, \lambda_1), L_t^*(x|\lambda_2), Z|\lambda_2) \\
&= G_t(x,\theta|\lambda_2) \qquad (6.17)
\end{aligned}
$$

The first inequality is because $S_t^*(v|\lambda_2)$ and $L_t^*(x|\lambda_2)$ are suboptimal pair of inventory and capacity when $\lambda = \lambda_1$. The second inequality is due to induction hypothesis. The last inequality is because $L(x,\lambda)$ is increasing in λ. □